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INTERVAL JUDGMENTS AND EUCLIDEAN CENTERS

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Summary*:* *We formulated the problem of finding a priority vector from an interval reciprocal matrix as an Euclidean center problem. The interesting result is that this formulation always has a solution and provides knowledge about the feasible region. The sign of the objective function of the Euclidean center formulation predicts the existence of a feasible solution that satisfies the constraints given by the interval reciprocal matrix. We showed that, if the Euclidean center objective function is positive, there are multiple plausible solutions, if it is negative, there no feasible solutions, and if it is equal zero, the feasible region consists of a single point.*

1. Introduction

Assessing priorities through the Analytic Hierarchy Process (AHP) involves elicitation of pairwise judgments concerning strength of preference (or dominance) among the subjects of comparison. This strength of preference is articulated by choosing a scale value from the 1-9 comparison scale (Saaty, 1980, 1986). An interesting problem is when the decision maker is unable to state his preference exactly due to, perhaps, uncertainty regarding the appropriate scale value to represent his strength of preference. Another case when an interval may be preferred to a single value is when a group of decision makers cannot reach consensus on a single scale value to represent their joint preference. In cases like these, the decision maker(s) may still have a preference "direction" in the sense that one element is preferred to the other but the strength of preference ranges over some scale values rather than be represented by a single one. Thus, in comparing elements *i* and *j*, preference (a_{ij}) may be stated through an inequality such as: $l_{ij} \le a_{ij} \le u_{ij}$, where l_{ij} and u_{ij} represent the lower and upper bounds, respectively, of preference taken as values from the 1-9 comparison scale. Filling a comparison matrix requires the elicitation of n(n-1)/2 entries before evaluation of the respective eigenvector is carried out. In this case, by analogy, one has to elicit <u>at most</u> (it is possible to elicit less, as will be discussed later) the same number of inequalities of the type shown above, we obtain the interval judgment matrix *IJ(A)*.

$$IJ(A) = \begin{pmatrix} 1 & [l_{12};u_{12}] & [l_{13};u_{13}] & \dots & [l_{1n};u_{1n}] \\ & 1 & [l_{23};u_{23}] & \dots & [l_{2n};u_{2n}] \\ & & 1 & \dots & . \\ & & & \dots & . \\ & & & & 1 \end{pmatrix}$$

Using intervals to express preference poses technical problems in processing these judgments to arrive at a representative preference structure. There have been two types of solutions to this problem: simulation (Arbel and Vargas, 1992; Saaty and Vargas, 1987) and linear programming (Arbel, 1991). The simulation approach assumed some sort of distribution (e.g., uniform) in the interval (l_{ij}, u_{ij}) , and generated a sample of size N of reciprocal comparison matrices, $(A_1, \dots, A_N)$. The principal right eigenvector of each matrix $\boldsymbol{w}_h$, h=1,…,N, is a sample point in the hyperplane $\boldsymbol{w}^T\boldsymbol{e} = 1$. Given the sample of eigenvectors it is possible to derive preference probabilities such as $P[\boldsymbol{w}_i > \boldsymbol{w}_j]$ and determine how the hyperplane $\boldsymbol{w}^T\boldsymbol{e} = 1$ is partitioned into preference regions. The linear programming approach provides solutions to the following problem:

$$\begin{aligned}
&Min\ w_0\\
&s.t.,\\
&l_{ij} \le \frac{w_i}{w_j} \le u_{ij},\ i, j = 1, 2, \dots, n\\
&w_1 + \cdots + w_n = 1\\
&w_i \ge 0,\ i = 1, 2, \dots, n
\end{aligned} \tag{1.1}$$

The solutions to this problem are the vertices of the feasible region. In (Arbel and Vargas, 1992) we showed that the average of the vertices coincides in the limit with the average of the simulation model. However, there are other ways of combining the vertices of the feasible region. One of them is using the concept of Euclidean center.

2. Euclidean Centers

Interval judgments generate a set of inequalities. Addressing this set of inequalities should ignore vertex information and address the inequalities directly. The Euclidean center is a point in space from which one inscribes the largest sphere inside the constraints whose intersection forms the feasible region. It should be noted, however, that Euclidean centers may not be unique and they do not exist for empty feasible regions. In these cases the set of inequalities are not solvable either, i.e., there exists no point that satisfies all the inequalities.

All existing methods use feasible points to generate a set of weights that satisfies the set of inequalities derived from the interval judgments. The feasible points are either interior to the feasible region or exterior, in which case the only ones available are the vertices of the feasible region. This point-wise depiction of preference information ignores the interval nature of the original preference statements. Constrained optimization problems are defined by a set of constraints defining a region in space. The specific region is, of course, affected by the nature of the constraints. The latter may be either linear or nonlinear, but the question of centers applies to both. Essentially, this question addresses the issue of how to inscribe an object within the region defined by the constraints. More than one centering approach can be defined which leads to a different inscribed object and a different derivation procedure for both the inscribed object as well as its center. In this paper we focus our main attention on the Euclidean center, and its uses in the Interval AHP.

The concept of a center has been a problem of some interest a few decades ago and has seen a rekindled interest in recent years. While it is important to many theoretical and applied problems—ranging from location theory to interior-point linear programming algorithms—specific literature on the subject is quite scant. The first publication that has treated the question of centers in an explicit manner is that of Huard (1967). In this paper, he develops a general algorithm for optimizing a concave function over a convex feasible region with the use of centers and bounds. The generality of the algorithm is maintained through the general definition of the distance that was used. However, due to the general formulation considered in that paper, only necessary conditions for distance were mentioned. Sonnevend (1985) defined an analytic center and used it to develop a linear programming approach based on Karmarkar's interior-point projective algorithm (Karmarkar, 1984). Boggs et al. (1989) used Huard's method of centers to enhance interior-point methods using dual affine trajectories. Fagan and Falk (1996) introduced a method of Euclidean centers for solving single-objective linear programming problems. Their work uses Euclidean center without identifying its origin, which seems to defy an original source. In a recent book by G.B. Dantzig, the idea of a Euclidean center is mentioned as an exercise to the student (Dantzig and Thapa, 1997, Ex. 6.1, p.151). Parenthetically we add that the suggestion in the book will *not* lead to the proper center.

The above references treat the issue of centers in an explicit manner. This issue appears in a less explicit manner in other areas as well. Location theory, for example, has traditionally been interested in establishing the best location for placing a service node in a given network (see, e.g., Hansen et al, 1987). While not addressed as a centering problem specifically, such a problem is exactly that of finding a center. One may be interested in finding a point that is as close as possible to all nodes in the network or, conversely, finding a node that is as far away as possible from all nodes in a network (for disposing of waste, for example). Such problems measure distances from nodes that form a network.

As seen from the above survey, more than one center has been developed in the past and more than one application has been identified (Sadka, 1998). Limiting our discussion to linear systems of constraints simplifies the way we derive these centers. It does not, however, reduces the possibilities available for consideration. A center can be defined in more than one way and each way leads to different analytic and geometrical implications. Specifically, we note that the intersection of a set of linear constraints for a bounded linear programming problem defines a polytope in *n*-dimensional space. When one talks about a *center* of such a polytope one usually refers to the point from which one inscribes some object contained by the polytope. The reasoning behind this operation, as well as the type of object used—be it a sphere, or an ellipsoid, for example— leads to different definitions for a center. When one wishes to be as *far* as possible from all facets of the polytope— which is equivalent to inscribing the largest sphere — we derive the Euclidean center.

3. Defining a Distance

Given a vector, $\boldsymbol{a}$, a hyperplane having this vector as its normal satisfies $\mathbf{a}^T\mathbf{x} = 0,$ for every point, $\boldsymbol{x}$, in the hyperplane. Translating the hyperplane (which, by definition, passes through the origin) we have a *linear variety* or an *affine* transformation, where the defining equation is now provided by $\mathbf{a}^T(\mathbf{x} - \mathbf{x}_0) = 0$. Arranging terms we arrive at the familiar expression for a hyperplane given by

$$\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_0 \triangleq \boldsymbol{b} \tag{3.1}$$

Projecting any vector, say $\mathbf{x}_1$, on another vector, say $\boldsymbol{q}$, is accomplished through a projection operator, **P**. The projected vector, $\hat{\mathbf{x}}_1$, is then given by

$$\hat{\mathbf{x}}_1 = \mathrm{P}\mathbf{x}_1 = \mathbf{q}\left(\mathbf{q}^T\mathbf{q}\right)^{-1}\mathbf{q}^T\mathbf{x}_1 \tag{3.2}$$

The discussion thus far is depicted in Figure 3.1 below.

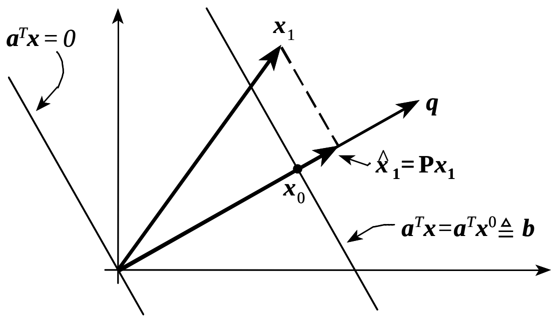

Figure 3.1. Projecting a vector on a normal to a plane

Comment:

The projected vector, $\hat{\mathbf{x}}_1$, provides the distance from the point $\mathbf{x}_1$, to the origin along the vector $\boldsymbol{q}$. To evaluate the distance from the point $\hat{\mathbf{x}}_1$, to the nearest point on the translated hyperplane we have to account for the translation, $\mathbf{x}_0$. This results in

$$\mathbf{d} = \mathrm{P}\left(\mathbf{x}_1 - \mathbf{x}_0\right) = \mathbf{q}\left(\mathbf{q}^T\mathbf{q}\right)^{-1}\mathbf{q}^T\left(\mathbf{x}_1 - \mathbf{x}_0\right) \tag{3.3}$$

which simplifies to

$$\mathbf{d} = \mathbf{q}\frac{\mathbf{q}^T\left(\mathbf{x}_1 - \mathbf{x}_0\right)}{\mathbf{q}^T\mathbf{q}} = \mathbf{q}\frac{\mathbf{q}^T\mathbf{x}_1 - \mathbf{q}^T\mathbf{x}_0}{\mathbf{q}^T\mathbf{q}} = \mathbf{q}\frac{\mathbf{q}^T\mathbf{x}_1 - b}{\mathbf{q}^T\mathbf{q}} \tag{3.4}$$

The *length* of this vector is provided by its Euclidean norm, which results in

$$\left\|\mathbf{d}\right\|_2 = \left\|\mathbf{q}\right\|_2 \frac{\left\|\mathbf{q}^T\mathbf{x}_1 - b\right\|_2}{\left\|\mathbf{q}\right\|_2^2} = \frac{\left|\mathbf{q}^T\mathbf{x}_1 - b\right|}{\left\|\mathbf{q}\right\|_2} \tag{3.5}$$

We can now use this result to find the distance from any given point to the subspace spanned by a given constraint. Denoting by $\mathbf{a}^i$ the i^{th} row ($1 \le i \le m$) of an $m \times n$ matrix, $\mathbf{A}$, the i^{th} boundary to the constraints polytope is then given by $\mathbf{a}^i\mathbf{x} = b_i$, which is one of the defining equations of the polytope in *n*-dimensional space.

The distance from a point $\boldsymbol{x} \in \boldsymbol{R}^n$ to a hyperplane described through $\mathbf{a}^i\mathbf{x} = b_i$, is given by

$$d_i = \frac{\left|b_i - \boldsymbol{a}^i\boldsymbol{x}\right|}{\left\|\boldsymbol{a}^i\right\|_2}, \quad \text{where} \quad 1 \le i \le m. \tag{3.6}$$

Dividing each constraint by its Euclidean norm, $\left\|\mathbf{a}^i\right\|_2$, we have $\quad d_i = \hat{b}_i - \hat{\mathbf{a}}^i\boldsymbol{x}, \quad \text{where} \quad \hat{b}_i = \frac{b_i}{\left\|\mathbf{a}^i\right\|_2}$ and $\hat{\mathbf{a}}_i = \frac{\mathbf{a}_i}{\left\|\mathbf{a}^i\right\|_2}$. Note that the definition in Equation (3.6) implies that every hyperplane divides the *n*-dimensional space into two halfspaces, one from each side of the hyperplane. All the vectors on the side which includes the origin, have a positive value for d_i (the distance from the hyperplane), and all the vectors on the other side have a negative value for d_i. In general, a set of linear constraints, $\boldsymbol{S}$, is defined through $\mathbf{S} = \left\{\mathbf{x} \in \Re^n \mid \mathrm{A}\mathbf{x} \le \mathbf{b}\right\}$. We see, therefore, that in case that a hyperplane of the set $\mathbf{S}$ is defined as $\mathbf{a}^i\mathbf{x} \le b_i$, the distance of a vector $\mathbf{x} \in \mathbf{S}$ from the hyperplane is calculated by $d_i = \hat{b}_i - \hat{\mathbf{a}}^i\boldsymbol{x}$. Similarly, in case of a hyperplane defined as $\mathbf{a}^i\mathbf{x} \ge b_i$, we use the formulation $d_i = \hat{\mathbf{a}}^i\boldsymbol{x} - \hat{b}_i$. After performing some simple calculations we have

$$\mathbf{x} \in \mathbf{S} = \left\{\mathbf{x} \mid a^i\mathbf{x} \le b_i\right\} \quad \Rightarrow \quad d_i = \frac{b_i - \mathbf{a}^i\mathbf{x}}{\left\|\mathbf{a}^i\right\|_2} \quad \Rightarrow \quad \mathbf{a}^i\mathbf{x} + \left\|\mathbf{a}^i\right\|_2 d_i = b_i \tag{3.7a}$$

$$\mathbf{x} \in \mathbf{S} = \left\{\mathbf{x} \mid \mathbf{a}^i\mathbf{x} \ge b_i\right\} \quad \Rightarrow \quad d_i = \frac{\mathbf{a}^i\mathbf{x} - b_i}{\left\|\mathbf{a}^i\right\|_2} \quad \Rightarrow \quad \mathbf{a}^i\mathbf{x} - \left\|\mathbf{a}^i\right\|_2 d_i = b_i \tag{3.7b}$$

Note, that using (3.7) for calculating the distances d_i, the condition $d_i \ge 0$ ensures that the vector $\mathbf{x}$, from which the distances are measured, is feasible (that is, satisfies the constraint no matter if it written as $\mathbf{a}^i\mathbf{x} \le b_i$ or $\mathbf{a}^i\mathbf{x} \ge b_i$). This is similar to the use of auxiliary variables—slacks or surpluses—when transforming an LP problem $\mathrm{A}\mathbf{x} \le \mathbf{b}$ or $\mathrm{A}\mathbf{x} \ge \mathbf{b}$ to its standard form $\mathrm{A}\mathbf{x} = \mathbf{b}$.

4. The Euclidean Center

Using the distance measure of a point to a constraint, we introduce now the concept of Euclidean center. Let $\mathbf{S}$ be the interior space of an *n*-dimensional polytope with *m* facets. A vector $\mathbf{x} \in \mathbf{R}^n$ is inside the polytope when $\mathbf{x} \in \mathbf{S}$. Assuming that the distance between $\mathbf{x}$ and the i-th facet, where $i = 1,...,m$, is $d_i(\mathbf{x})$, the Euclidean center of this polytope is found by solving the problem given by:

$$\underset{\mathbf{x}\in\mathbf{S}}{\text{Max}}\left\{\underset{1\le i\le m}{\text{Min}}\left\{d_{\mathrm{i}}(\mathbf{x})\right\}\right\} \tag{4.1}$$

This problem may be transformed into a linear programming problem by introducing an auxiliary variable α, which is maximized according to the following formulation

$$\begin{aligned} & \text{Max } \alpha \\ s.t. \quad & \mathbf{a}^i\mathbf{x} + \left\|\mathbf{a}^i\right\|_2 d_i = b_i \qquad i = 1,...,m \\ & \alpha \le d_i, \qquad i = 1,...,m \\ & d_i \ge 0, \qquad i = 1,...,m \end{aligned} \tag{4.2}$$

Example 4.1: Consider a set of linear constraints, ***S***, defined by

$$\begin{array}{llll} C_1: & x_1 + x_2 \ge 1 & C_4: & x_1 \ge 0 \\ C_2: & x_1 - x_2 \le 5 & C_5: & x_2 \ge 0 \\ C_3: & x_1 + 2x_2 \le 10 & & \end{array}$$

The Euclidean center of this region is obtained by solving the following LP problem:

$$\begin{array}{ll} \text{Max } \alpha & \\ \text{s.t} & \\ x_1 + x_2 - \sqrt{2}d_1 = 1 & \alpha \le d_1 \\ x_1 - x_2 + \sqrt{2}d_2 = 5 & \alpha \le d_2 \\ x_1 + 2x_2 + \sqrt{5}d_3 = 10 & \alpha \le d_3 \\ x_1 - d_4 = 0 & \alpha \le d_4 \\ x_2 - d_5 = 0 & \alpha \le d_5 \\ & d_1, d_2, d_3, d_4, d_5 \ge 0 \end{array}$$

whose solution is given by

$$\begin{pmatrix} \alpha \\ x_1 \\ x_2 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{pmatrix} = \begin{pmatrix} 1.910 \\ 1.910 \\ 1.910 \\ 1.994 \\ 3.536 \\ 1.910 \\ 1.910 \\ 1.910 \end{pmatrix}$$

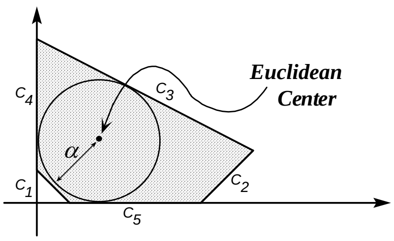

Figure 4.1. Euclidean center—computational method

The largest inscribed circle is of radius $\alpha = 1.910$ and is centered at point $[x_1, x_2] = [1.910, 1.910]$. The active constraints are C_3, C_4, C_5 with $d_3 = d_4 = d_5 = \alpha$.

Before we apply the notion of Euclidean center, we need to briefly mention some special cases encountered when deriving Euclidean centers of a polytope. Because the Euclidean center is derived as a solution of an LP problem, it may have one solution, infinite number of solutions or none at all (as in the case of an infeasible set). It may also result with an unbounded solution for an unbounded feasible set.

However, we note that this center always remains interior to the underlying polytope, and it is insensitive to redundant constraints. These properties are special to the Euclidean center. For example, the MinMax center, defined through solving $\operatorname*{Min}_{x\in S}\left\{\operatorname*{Max}_{1\le i\le m}\left\{d_i(x)\right\}\right\}$, may result on an edge of the polytope. The analytic center, defined through $\operatorname*{Max}_{x\in S}\prod_{1\le i\le m} d_i(x)$, may significantly change in the presence of redundant constraints.

5. The Euclidean Center of a reciprocal Matrix with Interval Judgments

Following the formulation outlined above, the MaxMin Euclidean center of the interval judgment problem is given by the solution to the following LP problem:

$$
\begin{aligned}
&Max\ \alpha\\
&s.t.,\\
&w_i - d_i = 0,\ i = 1,2,\ldots,n\\
&w_i - u_{ij}w_j + \sqrt{(1+u_{ij}^2)}d_h = 0,\ i,\ j = 1,2,\ldots,n,\ h = n+1,\ldots,n+\tfrac{n(n-1)}{2}\\
&w_i - l_{ij}w_j - \sqrt{(1+l_{ij}^2)}d_k = 0,\ i,\ j = 1,2,\ldots,n,\ k = n+\tfrac{n(n-1)}{2}+1,\ldots,n^2\\
&w_1 + \cdots + w_n = 1\\
&\alpha - d_i \le 0,\ i = 1,2,\ldots,n^2
\end{aligned}
\tag{5.1}
$$

while the MinMax Euclidean formulation is given by:

$$
\begin{aligned}
&Min\ \alpha\\
&s.t.,\\
&w_i - d_i = 0,\ i = 1,2,\ldots,n\\
&w_i - u_{ij}w_j + \sqrt{(1+u_{ij}^2)}d_h = 0,\ i,\ j = 1,2,\ldots,n,\ h = n+1,\ldots,n+\tfrac{n(n-1)}{2}\\
&w_i - l_{ij}w_j - \sqrt{(1+l_{ij}^2)}d_k = 0,\ i,\ j = 1,2,\ldots,n,\ k = n+\tfrac{n(n-1)}{2}+1,\ldots,n^2\\
&w_1 + \cdots + w_n = 1\\
&\alpha - d_i \ge 0,\ i = 1,2,\ldots,n^2
\end{aligned}
$$

Note that the difference between the two formulations is the set of constraints $\alpha - d_i \le 0$ in the MaxMin problem versus $\alpha - d_i \ge 0$ in the MinMax problem.

Example 5.1. Consider the simple 2-by-2 matrix:

$$
\begin{bmatrix} 1 & [2,4] \\ & 1 \end{bmatrix}
$$

The feasible region is given in Figure 5.1.

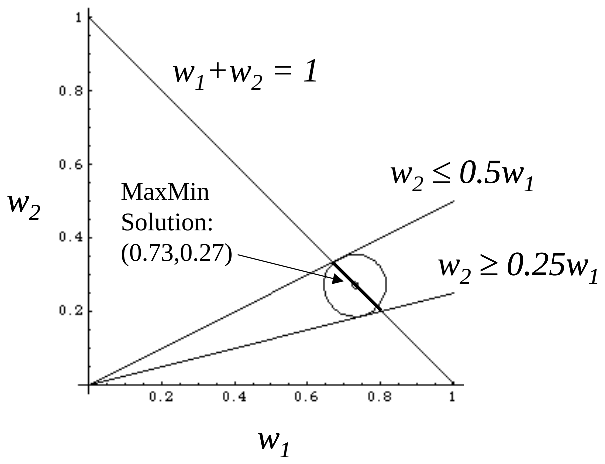

Figure 5.1. The MaxMin Solution

The MinMax solution is given in Figure 5.2.

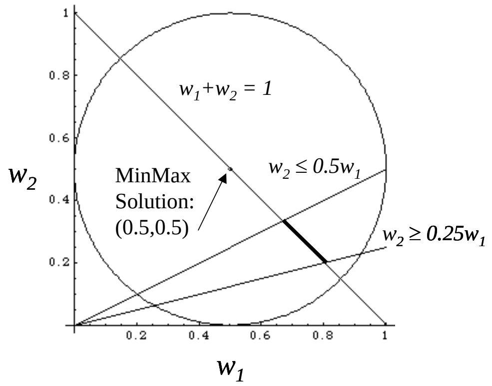

Figure 5.2. The MinMax Solution

Example 5.2. Consider the following interval judgment matrix:

$$\begin{bmatrix} 1 & [2,4] & [3,5] \\ & 1 & [1,2] \\ & & 1 \end{bmatrix}$$

The MaxMin problem is given by:

$$
\begin{aligned}
&Max\ \alpha \\
&s.t., \\
&w_i - d_i = 0, i = 1,2,3 \\
&w_1 - 4w_2 + \sqrt{17}d_4 = 0, \\
&w_1 - 5w_3 + \sqrt{26}d_5 = 0, \\
&w_2 - 2w_3 + \sqrt{5}d_6 = 0, \\
&w_1 - 2w_2 - \sqrt{5}d_7 = 0, \\
&w_1 - 3w_3 - \sqrt{10}d_8 = 0, \\
&w_2 - 1w_3 - \sqrt{2}d_9 = 0, \\
&w_1 + w_2 + w_3 = 1 \\
&\alpha - d_k \le 0, k = 1,...,9
\end{aligned}
$$

and the solution is given by:

$w_1 = 0.616,\ w_2 = 0.22,\ w_3 = 0.164$

$\alpha = 0.0396636$

$d_1 = 0.616, d_2 = 0.22, d_3 = 0.164, d_4 = 0.06, d_5 = 0.06, d_6 = 0.05, d_7 = 0.079, d_8 = 0.04, d_9 = 0.04$

The MinMax solution is given by

$w_1 = 0.3333,\ w_2 = 0.3333,\ w_3 = 0.3333$

$\alpha = 0.3333$

$d_1 = 0.333, d_2 = 0.333, d_3 = 0.333, d_4 = 0.24, d_5 = 0.26, d_6 = 0.15, d_7 = -0.15, d_8 = -0.21, d_9 = 0$

Note the MaxMin Euclidean center is a feasible solution equidistant from the non-redundant constraints. However, in general, it is not equidistant from the vertices of the feasible region. Solving the problems

$$
\begin{aligned}
&Max\ \ w_i \quad \text{or} \quad Min\ \ w_i \\
&s.t., \\
&w_i - u_{ij} w_j \le 0, i, j = 1,2,\dots,n \\
&w_i - l_{ij} w_j \ge 0, i, j = 1,2,\dots,n \\
&w_1 + \cdots + w_n = 1 \\
&w_i \ge 0,\ i = 1,2,\dots,n
\end{aligned}
$$

we obtain the vertices of the feasible region defined by the interval judgments. The number of vertices is at most $2n$. In the example at hand we have the vertices given in Table 5.1.

Table 5.1. The vertices of the feasible region in Example 5.2

	minw1	maxw1	minw2	maxw2	minw3	maxw3	average	MaxMin
w1	0.545455	0.689655	0.666666	0.571429	0.625	0.6	0.616367	0.616504
w2	0.272727	0.172414	0.166667	0.285714	0.25	0.2	0.224587	0.219775
w3	0.181818	0.137931	0.166667	0.142857	0.125	0.2	0.159046	0.163721

Note that the MaxMin Euclidean center is not the average of these vertices although it is close to it. Because the MaxMin solution is equidistant from all the preference constraints, it considers all the constraints equally important. The MinMax center does not always provide a solution that satisfies the interval preference constraints. So, for interval judgements one should consider the MaxMin Euclidean center.

A problem of interest is if the original interval reciprocal matrix yields a problem that does not have a feasible solution for (1.1), i.e., there does not exist a vector of priorities $(w_1, \dots, w_n)$ such that

$l_{ij} \le \frac{w_i}{w_j} \le u_{ij}$. In this case, should one say that there is no solution? Consider the following reciprocal matrix of interval judgments,

$$\begin{bmatrix} 1 & [2,4] & [\frac{1}{5},\frac{1}{3}] \\ [\frac{1}{4},\frac{1}{2}] & 1 & [1,2] \\ [3,5] & [\frac{1}{2},1] & 1 \end{bmatrix}.$$

There is no vector $(w_1,\ldots,w_n)$ such that $l_{ij} \le \frac{w_i}{w_j} \le u_{ij}$. However, the MaxMin solution exists and it is given by:

$$w_1 = 0.278,\ w_2 = 0.275,\ w_3 = 0.447$$
$$\alpha = -0.121864$$
$$d_1 = 0.278, d_2 = 0.275, d_3 = 0.447, d_4 = 0.2, d_5 = -0.12, d_6 = 0.28, d_7 = -0.12, d_8 = 0.184, d_9 = -0.12$$

Note, that in this case, the value of α is negative.

Let $\mathfrak{W} = \left\{ \mathbf{w} = (w_1,\cdots,w_n) \mid l_{ij} \le \frac{w_i}{w_j} \le u_{ij}, w_i > 0, \sum_{i=1}^{n} w_i = 1, i, j = 1,2,\ldots,n, \right\}$ be the feasible region of the interval judgment problem and let α^* be the optimal value of the MaxMin problem given by

$$\begin{aligned}
&\alpha^* = Max\ \alpha \\
&s.t., \\
&w_i - d_i = 0, i = 1,2,\ldots,n \\
&w_i - u_{ij}w_j + \sqrt{(1+u_{ij}^2)}d_h = 0, i, j = 1,2,\ldots,n, h = n+1,\ldots,n+\tfrac{n(n-1)}{2} \\
&w_i - l_{ij}w_j - \sqrt{(1+l_{ij}^2)}d_k = 0, i, j = 1,2,\ldots,n, k = n+\tfrac{n(n-1)}{2}+1,\ldots,n^2 \\
&w_1 + \cdots + w_n = 1 \\
&\alpha - d_i \le 0, i = 1,2,\ldots,n^2
\end{aligned}$$

Theorem: $\alpha^* \ge 0$ *if and only if* $\mathfrak{W} \ne \varnothing$

Proof: If $\mathfrak{W} = \varnothing$, then there exist *i* and *j* for which either $\frac{w_i}{w_j} > u_{ij}$ or $\frac{w_i}{w_j} < l_{ij}$. If $\frac{w_i}{w_j} > u_{ij}$, then since $w_i = u_{ij}w_j - \sqrt{1+u_{ij}^2}d_h$, there is an *h** for which $d_{h^*} < 0$, and because $\alpha^* \le d_h$ for all *h*, we have $\alpha^* \le d_{h^*} < 0$, and $\alpha^* < 0$. Likewise, if $\frac{w_i}{w_j} < l_{ij}$, then there is a *k** for which $w_i - l_{ij}w_j = \sqrt{1+l_{ij}^2}d_{k^*} < 0$ and $d_{k^*} < 0$ which implies that $\alpha^* < 0$.

If $\alpha^* \ge 0$, then because $\alpha^* \le d_h$ for all *h*, we have $d_h \ge 0$ for all *h*, and hence, $w_i = u_{ij}w_j - \sqrt{1+u_{ij}^2}d_h \le u_{ij}w_j$, for all *h*, and $w_i = l_{ij}w_j + \sqrt{1+l_{ij}^2}d_k \ge l_{ij}w_j$, for all *k*, and $\mathfrak{W} \ne \varnothing$.

Corollary: *If* $\mathfrak{W}$ *consists of one point then* $\alpha^* = 0$*, but the converse is not true.*

Proof: By the Theorem, $\alpha^* \ge 0$. In addition, if $\mathfrak{W}$ consists of one point then $u_{ij} = l_{ij}$, for all *i* and *j*, then $\frac{w_i}{w_j} = u_{ij} = l_{ij}$ for all *i* and *j*. Thus, we have $d_h = 0$, for $h = n+1,\ldots,n^2$, and $\alpha^* = 0$ follows. On the other hand, if $\alpha^* = 0$, then there are i^* and j^* for which $\frac{w_{i^*}}{w_{j^*}} = u_{i^*j^*} = l_{i^*j^*}$ and hence the corresponding d_{h^*} and d_{k^*} are equal to zero, but the remaining $d_h > 0$, for all $h \neq h^*, k^*$.

6. Conclusions

When the problem is consistent (interval judgments collapse into a single point) the eigenvector method and the Euclidean center coincide and alpha =0. When alpha > 0 there is an infinite set of consistent answers. When alpha < 0 the system is not solvable, i.e., the feasible region, defined by the constraints built on the interval judgments as inequalities in an LP model, is empty. The parameters d_i can be grouped into two sets: the first corresponds to the priorities, *w*, and the second, corresponds to the constraints (preference statements); the second group could be used to identify the most offending preference statement. Specifically, the largest absolute *d* from the second group could be used to identify the direction of change needed in those preference statements to improve the consistency measured (expressed) by the value of alpha. The objective of identifying this direction is to ensure that the system is solvable and hence by increasing alpha to a positive value solvability is achieved.

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