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AHP/ANP -- Where is Natural Zero?[1]

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Summary: A distinction of the Analytic Hierarchy/Network Process is that derived priorities are based upon ratio scales. The technique has been criticized on the grounds that ratio scales have a natural or absolute zero that AHP/ANP fails to recognize. This study investigates the role of natural zero in generating priorities. It concludes that zero is a demarcation point between positive and negative priorities, but does not play a central role in the actual generation of measures.

1. Introduction

Can a definition of something deprived of reality give rise to a lively concept?
(Robert Kaplan, The Nothing That Is, p. 63)

The Analytic Hierarchy Process (AHP) is distinguished from most other preference measurement systems by its use of ratio scales derived from redundant paired comparisons of objects, alternatives or events. The basic concepts were promulgated by Saaty in 1977 (Saaty, 1977) for decision making in hierarchical structures. Today, it has evolved into the Analytic Network Process (ANP) that structures problems as a system of clusters and nodes with interdependency and feedback (Saaty, 1996). ANP is a more general framework because it is capable of handling hierarchies as well as more complex systems.

In spite of its phenomenal success of in many practical decisions and with many other techniques, AHP/ANP has been under attack by persistent criticism from the academic community. Foremost amongst these is the issue of rank reversal (Belton and Gear, 1983; Dyer, 1990) associated with hierarchical structures that assume criteria to be independent of alternatives (Schoner & Wedley, 1989, Schoner et al, 1993). With the advent of ANP, this problem has diminished because network structures are capable of capturing dependent and independent relationships.

A controversy that persists is whether AHP/ANP priorities are truly in ratio form. Some claim they are not and dismiss the technique. For example, Martel and Roy in an otherwise credible analysis of multicriteria methods perfunctorily dismiss AHP with the following comment:

> *Nous laisserons notamment de côté la procédure AHP (cf.. Saaty, 1977, 1984) qui a donné lieu à de multiples controverses. À cet égard, la position de Barzilai (1988) mérite d'être citée: "A corollary is that procedures which measure preferences on a unit-scale in the absence of a universal or absolute zero (such as the Analytic Hierarchy Process) operate on an incorrect type of scale and consequently, produce incorrect results" (voir aussi Bana e Costa et Vansnick, 2001).*

Barzilai is perhaps the harshest critic of AHP. He (1998) correctly notes that ratio scales (or unit scales as he calls them) are scales where natural zero is fixed and the only free parameter is the unit. Among other things, he contends that the AHP's implied assumption of an absolute zero is insufficient and that the unit difference away from natural

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zero must be known before meaningful comparisons can be made. Saaty (1999) on the other hand, claims that AHP derives relative ratio scales and that a unit of measurement is not even needed.

In this paper, neither of those positions is accepted. Rather, the roles of natural zero and units of measure are explored. It is maintained that natural zero provides a demarcation point between positive and negative priorities, but it does not play a direct role in derivation of priorities. On the other hand, various intensity levels of attributes play a vital role. Those intensities are expressed as magnitudes from zero. Although notional in character, the resulting priorities do have a unit of measure. While many of Barzilai's criticisms have been challenged (R. Saaty, 2004), the existence and role of absolute zero has not.

2. A Short History of Natural Zero

In the development of our present number system, zero to imply nothingness, emptiness, null or naught arrived quite late. It is the last digit to come into use. Early concepts of numbers were properties of collections of objects. Zero and negative numbers do not have such properties and as a consequence their development was delayed. The Greek and Roman notation system, for example, has no provision for zero. Yet in spite of the absence of zero in Roman and earlier number systems, basic arithmetical operations of addition, subtraction, multiplication and division were still possible.

The first use of zero was as a positional indicator that allowed the extension of existing numeral systems. The Babylonians (3 B. C.) and Mayans, (4 A. D.) used different symbols (not "0") to denote a particular position being empty in a place-value notation system (i.e. just as zero denotes no hundreds and no tens in 2007). Although this enabled a symbol for nothingness to play an important information role, it did not involve zero as a number unto itself.

As a number that can interact with other numbers for mathematical operations, the greatest advancements were made by Hindu mathematicians, particularly Brahmagupta around 650 A. D. Besides developing many principles with regular numbers, he defined zero as the subtraction of a number from itself (n-n=0) and formulated other important relationships (n+0=n, n-0=n, -n+0=-n, nx0=0, etc.).

This acceptance of zero as a number was translated into Arabic, enabling further refinement by Islamic mathematicians. By 1100, the concepts had spread east to China and throughout the Arabic world into southern Spain. Although Greeks, Romans and medieval Western monks were familiar with an empty column on counting boards, they failed to conceptualize zero in written numbers. Meninger (1958) suggests this is because zero is something that must be there (as are units) in order to say nothing is there.

In 1120, an Englishman, Robert of Chester, translated into Latin the Arabic text of ibn al Khwarizmi who had earlier (7th century) translated the work of Hindu mathematicians (Flegg, 1983). Although this was the first of several translations, the main transfer of this knowledge to European societies is attributed to the Italian mathematician Fibonacci, also known as Leonardo of Pisa (Pogliani et al, 1998). In 1202, Fibonacci wrote *Liber Abaci* which explained and advocated the use of Arabic arithmetic and algebra.

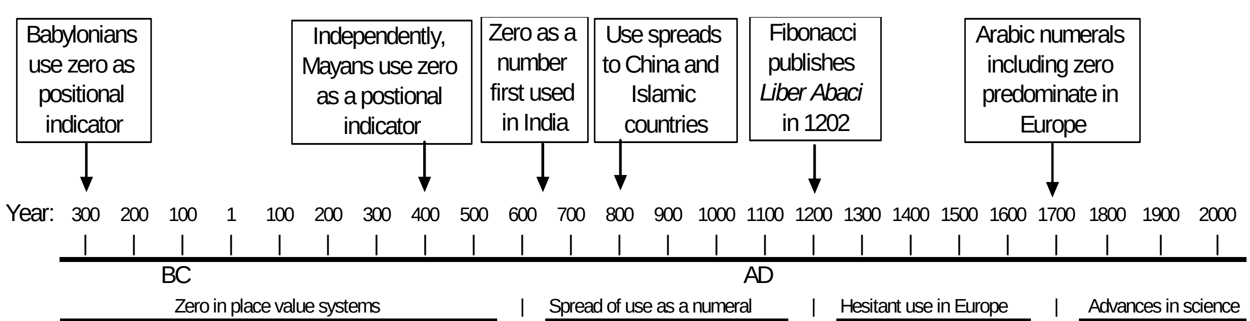

Figure 1: Timeline of the Development of Zero as a Number

Initially, Fibonacci's recommendation to use decimals, zero and the Hindu-Arabic numerals was met with considerable hostility and misunderstanding. Adoption slowly evolved as developments such as double entry booking (principle of zero balance), vanishing point in paintings (origin of zero), the use abstract concepts (algebra and paper money) came to the forefront (Rotman, 1987). By the early 17th century, Arabic numerals had completely replaced Roman ones as the dominant means for calculations. Thereafter, zero's usefulness became more evident for the development of calculus and advancements in the natural sciences.

Although the use of zero is commonplace today, there are still confusions that have persisted over the centuries. The first is when the 21st century started – technically, it was not until January 1, 2001, because historians did not use a year zero to begin counting (see Figure 1). The most common mistake, however, is the belief that n/0 = infinity. This is not the case. As Aristotle realized long ago (Flegg, 1983), it is not possible to divide a number by nothing. To realize this, simply reverse the process and ask yourself which infinite value multiplied by zero gives you that number.

Regarding the evolution of Arabic numbers and our decimal system, we should note that Fibonacci (who knew about zero) did ratio calculations without the use of zero. As well, many mathematicians prior to Fibonacci were able to perform ratio calculations without any knowledge of zero. For example, the Babylonians determined the ratio of the longest to shortest day (Kaplan, 2000) and troops of Alexander the Great employed ratio analysis for the construction and deployment of catapults. Flegg (1983) reports Aristotle had knowledge of zero but dismissed its use because zero could not be used in ratios and was of no use in his calculations to explain motion. Similarly, Archimedes formulated his law of the lever (Magnitudes are in equilibrium at distances proportional to their weights) and designed block and tackle pulley systems without the aid of zero. These facts should be an indication that maybe zero is not crucial for the derivation of ratio scales.

3. Types of Scales

Measurement can be done on different scales that represent different degrees of information and precision. Saaty (2004) describes the main types as follows:

> ***Nominal Scale****: invariant under one to one correspondence where, for example, a name or telephone number is assigned objects.*
>
> ***Ordinal Scale****: invariant under monotone transformations, where things are ordered by number but the magnitudes of the numbers only serve to designate order, increasing or decreasing; for example, assigning two numbers 1 and 2, to two people to indicate that one is taller than the other, without including any information about their actual heights. The smaller number may be assigned to the taller person and vice versa.*
>
> ***Interval Scale****: invariant under a positive linear transformation; for example, the linear transformation F = (9/5) C + 32 for converting a Celsius to a Fahrenheit temperature reading. Note that one cannot add two readings* x_1 *and* x_2 *on an interval scale because then* $y_1+y_2 = (ax_1 + b) + (ax_2 + b) = a(x_1+x_2) + 2b$ *which is of the form ax + 2b and not of the form ax + b. However, one can take an average of such readings because dividing by 2 yields the correct form. Putting it differently, one cannot add two temperature measurements to obtain an outcome that corresponds to a physical situation in which the two objects being measured are added. For example, 10 degrees of temperature added to 15 degrees of temperature do not make 25 degrees of temperature. At best their sum might be 15 degrees.*
>
> ***Ratio Scale****: invariant under a similarity transformation,* $y = ax,\ a > 0$*. An example is converting weight measured in pounds to kilograms by using the similarity transformation K = 2.2 P. The ratio of the weights of the two objects is the same regardless of whether the measurements are done in pounds or in kilograms. Zero is not the measurement of anything; it applies to objects that do not have the property and, in addition, one cannot divide by zero to preserve ratios in a meaningful way. Note that one can add two readings from a ratio scale, but not multiply them because* $a^2x_1x_2$ *does*

not have the form ax. The ratio of two readings from a ratio scale such as 6 kg/ 3 kg = 2 is a number that belongs to an absolute scale that says that the 6 kg object is twice heavier than the 3 kg object. The ratio 2 cannot be changed by some formula to another number. Thus we introduce the next scale.

***Absolute Scale**: invariant under the identity transformation x = x; for example, numbers used in counting the people in a room. An absolute scale is a special instance of a ratio scale with the constant multiplier equal to one. The real numbers that all mathematics uses to solve equations belong to an absolute scale. The natural numbers and the real numbers are defined in terms of 1-1 correspondence and equivalence classes, not in terms of laying out a unit of measurement starting with an origin.*

The first 4 of these are the standard hierarchy of measurement scales defined by Stevens (1946). The last scale is absolute since its character is quite independent of any physical attribute of a specific object that could be treated as the unit. In this sense, it is a fundamental scale.

Barzilai (2005) takes a different approach to defining scales. He classifies measurement scales by the mathematical operations that are enabled on the resultant scales. A proper scale enables the operations of addition, subtraction, multiplication and division. Amongst proper scales are strong ones that enable limit operations of calculus. Scales that do not have the arithmetic operations are termed weak.

4. Defined vs. Derived Scales

When referring to scales, it is important to distinguish between those that are defined versus those that are derived. By a defined scale, we mean one that has zero and a well-defined unit established in advance. Such scales with standardized values can be used to directly measure any object that has that dimension. Temperature, for example can be measured on established Fahrenheit or Celsius scales (integer scales) or the Kelvin scale (ratio scale).

Barzilai (2005) contends that absolute zero and a defined scale must be known before ratio measurement is possible. Since absolute zero is undefined in AHP methodologies, he claims that it does not use valid ratio scales.

Scale ratios are defined for vector (linear) scales but are undefined for affine-scale values. The example of temperature, where the ratio T_1/T_2 of two temperatures became well defined only when the existence of an absolute zero for the property under measurement is established. In the case of subjective properties (non-physical properties such as beauty of the objects being compared), the existence of an absolute zero has not been established. As a result, scale ratios for preference are undefined. The classical theory of measurement and mathematical psychology failed to recognize that measurement models of subjective properties that are based on ratios of scale values are not valid (see e.g. Coombs et al, 1970). In particular, the AHP (see Saaty, 1977) incorporated this flaw and for this reason is not a valid methodology.

Here, it is clear that he is referring to defined scales. He contends that because absolute zero cannot be identified in advance for subjective properties, affine scales with an arbitrary zero must be used. He further contends that classical measurement theory uses weak scales (addition and multiplication not possible), because units are unknown or undefined Barzilai (2005).

A derived scale is different. It parameters are not known beforehand. Instead, the scale values are generated from a questioning process and mapping that evaluates objects possessing the dimension. This is what Saaty (2004) was referring to in his paper on "Scales from Measurement – Not Measurement from Scales".

The basic building blocks for deriving an AHP/ANP scale are paired comparisons. Even though Figure 2 shows the process with 3 alternatives, the simplest example is provided by just two alternatives and one comparison. Assuming the comparison is accurate, one can derive the ratio scale between the two items by evaluating their intensities. Although not always done in two stages, the comparison is derived from the following questions:

1. Which object, A or B, dominates the other? (an ordinal question)

2. By how many times does that object dominate the other? (a ratio question)

In a paired comparison matrix, the less dominate object from the first question becomes the denominator and unit of measure for the comparison. In the second question, the intensity for the more dominant object is determined. As shown in Figure 2, the common dimension of the objects cancels out; leaving a measure that is an absolute value that has an isomorphic relationship to the fundamental absolute scale. Since that fundamental counting scale has a dimension of one, the absolute values entered into the paired comparison matrix are dominance units of the row alternative over the column alternative. This means that the column objects in Figure 2 are the units for the absolute values in the columns. We should note that since the paired comparison value belongs to an absolute scale, the only permissible transformation is multiplication by one, the identity value. This is not the same as a regular ratio scale where values can be multiplied by any positive number.

As shown in Figure 2, Saaty's fundamental uses the odd values from 1 to 9: 1=equally, 3=moderately, 5=strongly, 7=very strongly, 9=extreme. Entries can be made according to numeric, verbal, graphic or questionnaire modes with provision for even or decimal values between any of the markers. Although values >9 are possible, they would be unusual, because the structuring procedures for AHP/ANP suggest that compared items should be within one order of magnitude (accuracy suffers if the compared objects are too dissimilar). If the derived scale must exceed one order of magnitude, this can be achieved by linking successive clusters that contain objects spanning the different magnitudes.

In Figure 2, A dominates C (answer to question 1) by 6 times (answer to question 2). The absolute comparison value on the fundamental scale is A/C=6/1. This absolute value is the number of counts or the factor by which B is multiplied to reach A (an Archimedean concept of balance). By the reciprocal relationship, C/A=1/6. Since choice of unit is arbitrary, different relative priorities could be A=6, C=1 (C is the unit); A=1, C=1/6 (A is the unit); A=0.857, C=0.143 (a concatenation of A and C values is the unit). Notice that the concatenation is of the represented absolute values, not the objects themselves.

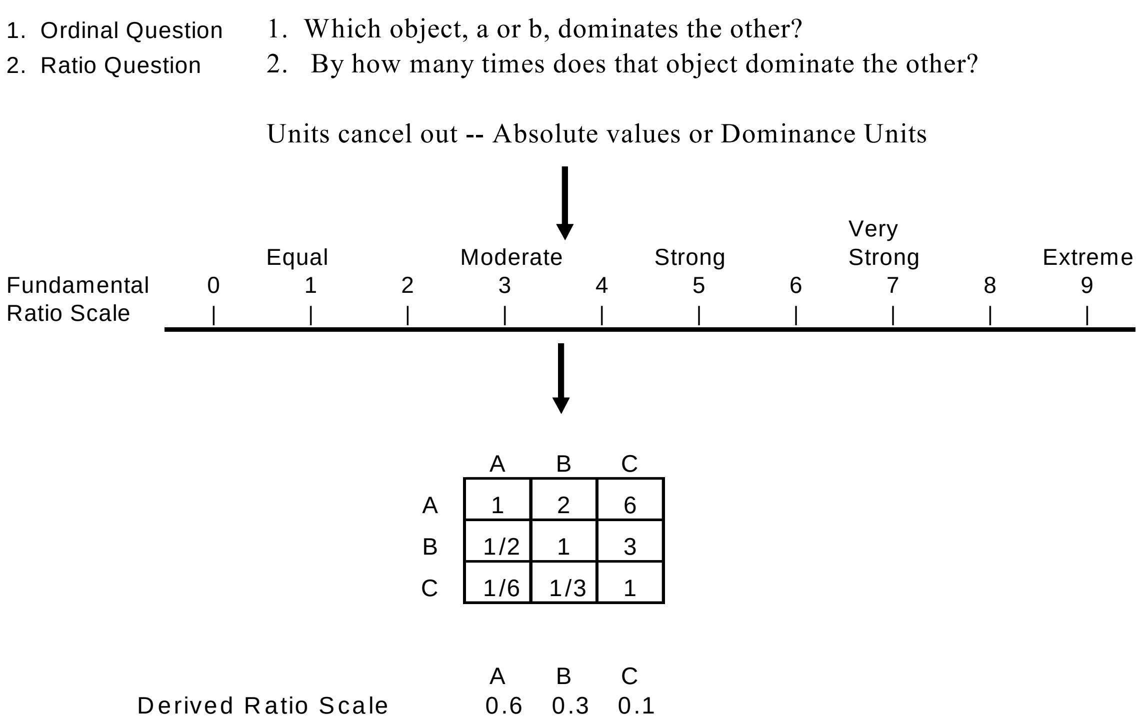

Figure 2: From Fundamental to Derived Scales

For problems with n>2 objects, we pair compare the intensities of the various objects available to measure. This requires an enlarged paired comparison matrix with n(n-1)/2 of n^2 comparisons. The remaining comparisons are not required because the diagonal values will be unity and the reciprocal values are derived from $a_{ji} = 1/a_{ij}$. Every number except the additive identity, 0, has a multiplicative inverse. Thus, zero is not a possible paired comparison value, even though we may use it in the matrix to denote no comparisons. Provided the diagonal is supplemented with an extra value for each zero appearing in the row, the derived solution with incomplete comparisons will not be distorted. (Takeda and Yu, 1988).

Although all the paired comparison values in Figure 2 are absolute numbers, each column has a different interpretation. The object listed at the top of each column is the unit of measure for that column. For the first column, for example, the unit is object A, the object that has the greatest attribute value. With it as the unit, all other objects take lower absolute values (1/2 for B and 1/6 for C). In the last column, object C is the unit. Having the least attribute value, all other objects take absolute values greater than the unity (6 for A and 3 for B). Viewed from this perspective, each column is like an absolute scale from different parts of the fundamental scale. The only permissible transformation of these absolute values is multiplication by unity.

With consistency (as exists in Figure 2), we can normalize any column of the paired comparison matrix to determine the derived ratio scale. For example, we can multiply the last column by 1/10 to get the derived ratio scale. Notice that the normalization (multiplication by the inverse of the column sum) is not a permissible transformation of absolute numbers. However, multiplication by a constant is a permissible transformation of a ratio scale, of which the absolute scale is a special case. Accordingly, the normalization transforms the absolute values into a derived scale of relative ratio values. In essence, the representation on the fundamental scale has been transformed into measurement on the derived scale. This is in keeping with the representational approach to measurement whereby numerical representations are derived from a fundamental scale (Michell, 1999).

An issue regarding the derived ratio scale is its unit of measure. Since the normalized scale values in Figure 2 are not relative to any one of the objects, it is possible to say that there is no unit of measure. As Saaty (2006, p. 558) states: "Derived relative scales need not have a unit, but they can after they are derived by dividing by the value of one of them if desired." An alternate perspective is that all ratio scales are relative to something and that something is the unit. For the derived ratio scale in Figure 2, the unit is a notional object represented by the concatenation of the absolute values of the measured objects. Notice in the normalization of the third column that 10 is the absolute value of the concatenation values of A, B and C. A hypothetical object representing this concatenated absolute value is the unit for the relative values 0.6, 0.3 and 0.1. That principle applies irrespective of which column is chosen for the concatenation of absolute values. Notice that the concatenation is of the represented absolute values, not the objects themselves. A notional object that represents the concatenation of the actual objects would not necessarily lead to the same ratio between the objects.

A remaining issue is the place and role for zero on the fundamental and derived scales -- the usual AHP/ANP practice is to exclude zero from the fundamental and derived scales. This is sensible given that no object can take on the value of zero. On the other hand, including zero in the scale would remind the user that the origin of the intensity is a hypothetical object that represents nothingness. We say a hypothetical object, because no comparison can be divided by zero; in other words, we never compare a real object to a nothingness object. On the Kelvin scale, for example, scientists have come close to, but never achieved, absolute zero for heat energy that has been hypothesized and calculated as the endpoint for the temperature scale.

This process of deriving measurements from compared objects is quite different from measuring objects one at a time according to a predetermined scale that has a known unit and zero. With derived measures, there are no prior values for mapping objects to a scale and there is no pre-specification of an explicit unit or the zero value (Saaty, 2006). Rather, the scale measures are determined from the objects themselves when they are treated as different units of measure. Once a scale is derived for the objects, it is possible to specify a unit just as was done above for the single comparison example. As well, it would be possible to show zero as the origin of the scale. Unfortunately, most AHP/ANP scales omit the zero point and leave the unit undefined. As we shall see with negative priorities, the specification of the zero point becomes imperative.

5. Distances, Differences and Intensities.

Although Saaty (2004) notes that there is no requirement for a scale to have a unit or an origin with zero value (e.g. an ordinal scale), the usual definition of a ratio scale is that it has an arbitrary or discretionary unit, but a meaningful and specific zero point. A ratio scale is the number of units a number is from zero. An interval scale, on the other hand, has both a discretionary zero and a discretionary unit.

The selected unit in a ratio scale defines equidistant points for the scale. In the ratio formula y=ax, "x" is the number of units away from zero. To get "y", "x" is multiplied by a, the unit, which maintains a constant value of 1. Hence, y= (1)x =x. Since 1 is the identity transformation for multiplication, Saaty (2004) classifies "x" as belonging to an absolute scale. In addition, Saaty refers to "x" as being dimensionless because any pre-existing units of measure would cancel out in comparisons (e.g. 6kg/3kg = 2). Rather than say "dimensionless", perhaps a clearer definition would be that "x" is a quantity with a dimension of 1. The absolute value "x", the counting number, still measures the number of units used to describe the scale (e.g. a unit of 3kg in the previous comparison). In other words, the quantity y is defined as the product of the absolute number 2 and one unit (the 3kg object) that is used as the standard of measure. This creates the Archimedean balance, 6kg = 2(3kg), which is shared by all ratio scales (Michell, 1999). Just as in counting, 2 is a number that exists unto itself, but to give it meaning we have to ask "two" of what? In this case it is two units (2/1) away from zero where the unit happens to be a 3kg object.

If a single comparison is used to derive a ratio scale, then that comparison must be the pure number that is counting the intensity or number of dominance units each object is away from natural zero. That is an assumption that is made when comparisons are made. Seldom, however, is that assumption made explicit to the person making the comparison. In the case of preference elicitation, the person would have to be sensing the intensity from no preference.

For subjective scales, Barzilai (2005) is unwilling to accept the assumption that people express their preferences by intensities away from zero. He contends (2001, p. 4) that the need to determine the existence of an absolute zero has long been recognized as a requirement for ratio scales (von Neumann and Morgenstern, 1951). Citing Stevens (1946, 1951), he states:

> *More explicitly, Stevens [20, p. 679] states: An absolute zero is always implied, even though the zero value on some scales (e.g. absolute temperature) may never be produced" and "If, in addition, a constant can be added (or a new zero point chosen), it is proof positive that we are not concerned with a ratio scale" [21, p. 29]. Since an absolute zero has not been established (and in all likelihood, does not exist) for preference measurement, preference cannot be measured on ratio scales.*

With respect, the p. 29 quotation of Stevens (1951) may be a misconception. Stevens was describing how to determine the type of measurement. He stated that if a consistent set of rules cannot be used to determine the type of scale, then

> *… we may seek the final and definitive answer in the mathematical group structure of the scale form: in what ways can we transform its values and still have it serve the functions previously fulfilled. We know that the numerical values on all scales can be multiplied by a constant, which changes the numerical size of the unit (unless the multiplier is unity itself). If, in addition, a constant can be added (or a new zero point chosen), it is proof positive that we are not concerned with a ratio scale"*

The key here is whether the transformed scale can still fulfill its former functions – whether the transformation leaves the scale form invariant. For the scale type to be maintained, the transformation should not change the equality of ratios for a ratio scale, the equality of differences or intervals for an interval scale, the increasing monotonic relationship for an ordinal scale and the one to one substitution for a nominal scale. Ratio scales satisfy all these functions and multiplication by a constant would leave it and other scales invariant as to their properties. Similarly, addition to an interval scale would leave it and other non-ratio scales invariant with respect to their required properties. However, adding a non-zero constant to a ratio scale would destroy the equality of ratios and make it interval measurement. That is not proof that the former scale was not ratio to start with. It is only proof the transformed scale is interval at best.

Since Barzilai (2005) contends that absolute zero cannot be identified for subjective properties, he proposes the use affine scales with an arbitrary zero. This solution, like that of Salo and Hamalainen (1997), is to use preference differences from affine scales. The result is an interval answer that is suitable to "best choice" decisions where just an ordinal answer is sufficient. But, it is not suitable for more complex decisions such as resource allocation where ratio answers are necessary.

If an affine transformation is applied to a ratio scale, differences between any two objects would still be the same and ratios of their unit distances would still form the same ratio. For example, consider the two objects, A and B, that weigh 3kg and 6kg respectively (depicted as intensity lines in Figure 3). If y of the ratio scale is transformed by y'=ax+ß, where arbitrary ß = -2, what was formerly 2 on the ratio scale would be now become the arbitrary zero. The 2 objects, however, have not changed weight. The differences of the two objects would still be 6-3= 4-1 = 3 on both scales and the ratio of their distances B/A = (4-(-2))/(1-(-2)) = 6/3 = 2 would also be the same.

As Stevens (1951, p. 27) has pointed out, "Differences between values on an interval scale become ratio scale measures for the simple reason that the process of determining a difference (i.e. subtraction) gets rid of the additive constant β". In order to take a ratio of differences, however, we would need to measure from a third reference alternative as suggested by Salo and Hamalainen (1997). With a referent object C for Figure 3 that weighs 1 kg on the ratio scale and -1 on the interval scale, the ratio of such (B-C)/(A-C) differences from C would be (6-1)/(3-1) = 5/2 = 2.5 on the ratio scale and (4-(-1))/(1-(-1)) = 5/2 = 2.5 on the interval scale. The ratios of differences are the same (and indicate B>A), but the correct ratio between B and A for AHP/ANP purposes is 2/1. Preference differences on an interval scale only correspond to the ratio scale when the referent alternative coincides with natural zero (Schoner et al, 1997).

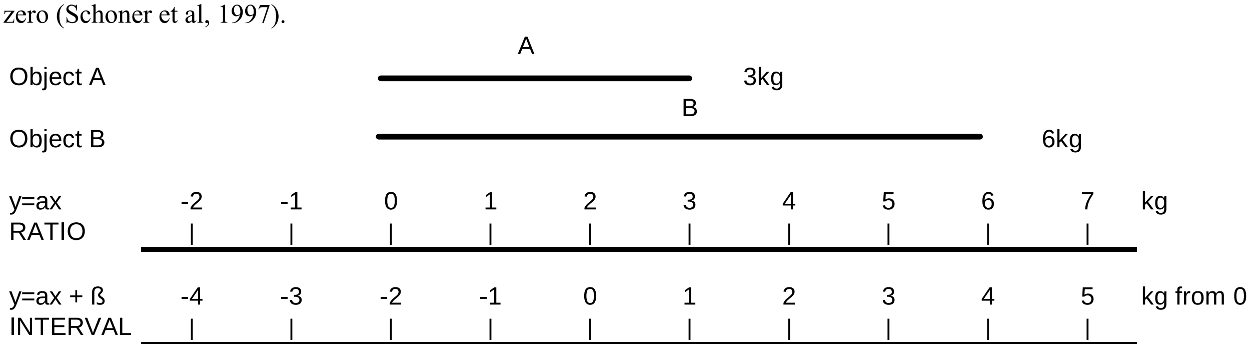

Figure 3: Ratio and Interval Measurement of Two Objects

Perhaps the confusion over this issue use of preference differences comes from the distance from Philadelphia example that Salo and Hamalainen (1997) cited. In that example (Saaty, 1980), Philadelphia was used as the common reference point from which distance was measured. Distance (length) could be considered an interval measure where zero could be placed arbitrarily any point on the line. In the Philadelphia case, the reference alternative, Philadelphia, is ground zero, exactly coinciding with natural zero. As stated by Von Neumann and Morgenstern (1941, p. 22), "Distance has a natural zero – the distance of any point from itself." Thus, that example is one of the special cases where the referent for differences coincides with natural zero.

What concerns critics of AHP is the possibility that people sense intensity from an origin other than absolute zero. For example, consider a person who is comparing two golf bags, A weighing 3 kg and B weighing 6 kg. While bag B is twice the weight of bag A, the DM may think that the lightest possible golf bag is one that weighs 2 kg. If the DM measures intensities from this referent minimum rather than absolute zero, the ratio will be one of unit differences from 2 kg: (B-2)/(A-2) = 4/1. Without firm guidance that comparison intensities should be from absolute zero, AHP critics claim that AHP/ANP does not generate ratio scales.

6. Negative Priorities

In the Philadelphia example, we should note that some cities can be in opposite directions. On the straight line in Figure 3, this would be equivalent to a city being on the negative side of zero. When comparing such cities, the absolute value of the comparison would be put into the paired comparison matrix. For the Philadelphia example, this would not be a problem, because only distance, not direction, is of interest. In many problems, however, a negative value signifies detraction from the criterion of interest. This occurs when we have criteria such as cost, risks, losses, or the weight of a golf bag. Natural zero plays a central role when evaluating with negative criteria

Like the history of negative numbers, the recognition of negative values for AHP priorities occurred relatively late. Although Saaty reports that he used negative priorities for difference comparisons in 1976 and for scenario construction in 1983 (Saaty and Ozdemir, 2003), the first person to formally recognize and advocate the use of negative priorities was Schoner (1996). Later, Saaty and Ozdemir (2003) and Millet and Schoner (2005) analyzed negative priorities in a more comprehensive manner.

When all alternatives or objects reflect negative attributes, the traditional AHP/ANP procedure is to use the absolute values of negative dominance to generate a set of positive priorities (larger priority signifies greater negativity). Then, the priorities signifying greater negativity are converted to positive attributes by simply normalizing the inverse of the priorities (formerly large priorities representing large negatives are transformed into small positive priorities). Saaty and Ozdemir (2003) claim this simple process is justifiable and legitimate

Millet and Schoner (2005), on the other hand, looked at this process and came to the opposite conclusion.

> *Such an inversion process amounts to dividing a ratio scale measurements by a negative value even though only division (or multiplication) by a positive constant is a permissible transformation. Furthermore, the value we are dividing by is not even a constant since each value gets divided by the negative of its own squared value. The effect is not only an arbitrary shifting of the natural zero point below values that should be negative, but also a reversal of strong and weak effects. Large costs become small positive effects while small costs become large positive effects. Hence, while the AHP claims it produces value measurements on a ratio scale, its value inversion procedure leads to a clear loss of the ratio scale property.*

Although this process does not involve dividing by negative priorities (all generated priorities are positive), the effect is to divide by the square of its value. For example for the first priority, the transformed priority $\hat{p}_1$ is

$$\hat{p}_1 = \frac{\frac{1}{p_1}}{\sum_{i=1}^{n}\frac{1}{p_i}} = \frac{p_1^2\frac{1}{p_1}}{p_1^2\sum_{i=1}^{n}\frac{1}{p_i}} = \frac{p_1}{p_1^2\sum_{i=1}^{n}\frac{1}{p_i}}$$

Since p_1^2 in the denominator will be different for each p_i, the transformation is not a division (or multiplication) by a positive constant.

A numerical example will help to resolve this issue. Table 1 gives values of benefits and costs in terms of dollars. Notice that alternative C is unprofitable: it has a negative net benefit and that its B/C ratio is less than one. Alternative B is best because it has the highest net benefit and B/C ratio.

Table 1: Example Data of Benefits and Costs

Alternatives	Benefits	Costs	Net Benefit	B/C Ratio
A	$ 2,000	$ 1,000	$ 1,000	2.0
B	$ 6,000	$ 2,000	$ 4,000	3.0
C	$ 4,000	$ 5,000	$ (1,000)	0.8

Table 2 expresses the dollar amounts in terms of priorities (Bp, Cp and 1/Cp). Notice under Cp that C, with the largest cost, has the largest cost priority. Notice too that the inverse transform (column 1/Cp) changes that largest cost priority to the smallest one. As transformed, these inverse priorities are supposed to be treated like positive criteria, to be added to other positive criteria.

In using priorities to synthesize composite results, we must remember that they are relative to each other, but they are no longer in commensurate units as they were in dollars. Accordingly, we must get commensurability by weighting them. Since the ratio of benefits to costs is $12,000/$8000, the correct weights are 0.6/0.4. These weights are used to get the composite results in the last three columns.

Table 2: Priority Results related to Benefits and Costs

Alternatives	Wb=0.6 Bp	Wc=0.4 Cp	Wc=0.4 1/Cp	Weighted Bp - Cp	Weighted Bp + 1/Cp	Weighted Bp/Cp
A	0.167	0.125	0.588	0.05	0.335	2
B	0.500	0.250	0.294	0.2	0.418	3
C	0.333	0.625	0.118	-0.05	0.247	0.8

We can see in Table 2 that the Weighted Bp-Cp and Weighted Bp/Cp give the results that emulate the true answers in Table 1. In contrast, the Weighted Bp + 1/Cp result does not give the correct answer. As Millet and Schoner caution, inversion loses the ratio relationship between positive and negative criteria. We should follow their advice to not use the inversion process.

Millet and Schoner (2005) call their solution for incorporating negative values the bipolar AHP. After generating relative priorities for positive and negative criteria, they renormalize so that the best positive alternative has a value of +1 and the worst negative alternative has a value of -1. These extreme alternatives become the bipolar anchors, units and links that are compared for determining criteria weights. The absolute value of the extreme alternatives under each criterion is used in the comparison of how far each departs from natural zero. Since the extreme alternatives are the units of each criterion, their relative criteria weights can be passed to themselves and other alternatives to determine composite priorities. This same principle with absolute values can be used for a single criterion that has both positive and negative alternatives. As an alternative to the absolute value procedure, Millet and Schoner suggest a gradient value method that asks decision makers to compare the value of common percentage improvement in the extreme alternatives.

Saaty and Ozdemir (2003) also use absolute values, but in a slightly different manner. First, they derive composite priorities for all the positive and negative criteria. This is equivalent to synthesizing separate hierarchies of positive and negatives to get a single criterion for each. Criteria weights for each are then determined by the ratio of the sum of synthesized priorities of the alternatives to the absolute value of the difference between the sums of the values of the alternatives with respect to each of the synthesized priorities. Since the absolute difference of the sums of values of alternatives is the same as the sum of the two absolute differences away from zero, the process is normalizing the totalities in one direction to the sum of totalities in both directions. The same result could be achieved by comparing the totality of the alternatives in one direction to the totality of alternatives in the other direction.

7. Definition of Zero

So, we know that natural zero is the demarcation point between positive and negative values. We also know that zero is the origin for sensation of intensities for paired comparisons. But what is the exact definition of natural zero.

Millet and Schoner (2005) expressed concern that standard AHP fails to recognize and define the true zero preference point. They noted that a natural zero is "…a zero that, as its name implies, signifies a *lack of magnitude*". With that concept in mind, they defined natural zero as the descriptive value of an alternative that neither adds nor subtracts from the overall desirability of the alternative. In other words, it is neither positive nor negative – it is zero.

Notice their use of the terms “lack of magnitude” and “neither adds nor subtracts”. These concepts differentiate natural zero from an arbitrary zero. While it is possible for an arbitrary zero to be natural zero, it is unlikely, because the discretionary zero point generally has some magnitude. That is what differentiates a ratio scale from an interval scale. In Figure 3, for example, the zero point on the interval scale coincides with an intermediate magnitude for both object A and B.

My colleague, Eng Choo, describes natural zero as the “absence of attribute intensity”. As he points out, “…we do not need to have an object with measure zero as in "there is no rock of zero weight". If we can accept "total absence of weight", we should be able to accept "absence of attribute intensity" as well.”

Saaty and Ozdemir (2003) take a very different interpretation of natural zero.

> *Relative measurement in the AHP is derived measurement. Its zero is not absolute but relative to the goal of the specific decision. If one compares stars according to size, their zero is different than the zero comparing atoms according to size. The zero used to measure atoms and stars on a physical scale is the same absolute zero. An absolute zero requires a unit of measurement. A relative zero does not. To make a relative zero a more general kind, we need to compare atoms with stars according to size so their two relative zeros are combined like all the other numbers into a new zero.*

This is a somewhat startling definition with many implications. If relative zero is not absolute zero, then what is it? Is this relative zero different from the natural zero that demarks the change point between positive and negative priorities? When a paired comparison is made, is the origin for dominance or intensity from natural zero or is it from some other relative zero? And if paired comparisons are absolute values from a fundamental scale, is not natural zero part of the numbers on that absolute scale? Finally, how is relative zero differentiated from the arbitrary zero of an interval scale? Perhaps Barzilai has justification in saying AHP fails to identify natural zero and measure from it.

It seems to this observer that both physical and derived ratio scales have natural zero as the origin and the only arbitrary parameter is the unit of measure. Whereas an absolute scale has zero origin and a dimension of one, absolute zero by itself does not have nor need a dimension. Zero is zero – it has no intensity or dominance. It represents the absence or lack of a real thing. And representing nothing, it cannot be used as a unit, it cannot be compared to itself as can real objects, and it cannot be used in ratio comparisons. Since zero is the origin for every ratio value, every value on the scale relates to zero. Hence, a unit of measure requires zero, but not vice versa. As shown in Figure 3, this applies to both interval and ratio measurement.

Another curiosity about the Saaty and Ozdemir statement is how zero in a relative ratio scale can differ from natural zero in a physical ratio scale. Perhaps Saaty’s definition of an absolute scale gives us a clue. As described above, he seems to say that an absolute scale does not have unit and is not measured from an origin. This coincides with the historical perspective of natural numbers where only positive integers (without zero) were recognized as real. It also coincides with the concept of dimensionless paired comparisons being devoid of a unit. Perhaps the resulting vector of relative measurements inherits the dimensionless nature of the comparisons and the absence of an origin.

An alternate interpretation is possible if the origin of the absolute scale is assumed to be zero (i.e. all non-negative numbers). As such, both the intensities and dominance in comparisons would be relative from absolute zero. When dominance values are placed in the paired comparison matrix, it is quite clear that the unit of each column is the j^{th} object at the top of the column. When those columns of units are converted to a vector of relative values, it is hard to imagine how the common absolute zero is lost as the origin of the scale. A more logical conclusion is that absolute zero is the origin of both paired comparisons and the derived relative scale. While the relative ratio scales of atoms and stars will be in different units, they will both have the same absolute zero as the origin.

In Figure 4, the objects A and B are analogous to small objects like atoms while objects C and D are larger objects like stars. The relative ratio values of A and B are 0.333 and 0.667 respectively. Notice that these relative ratios are equivalent to a proportional transformation of the kilogram scale so that the values lie between absolute zero and 1.

In effect, the unit of the relative values is equivalent to a hypothetical object that is the sum of the absolute values of A and B. The same applies to the relative values of objects C (0.429 and D (0.571) – the kilogram scale is transformed to lie within 0 and 1 and the unit is equivalent to a hypothetical object that is the sum of the absolute values of C and D. Notice that although each relative scale is compressed to sum to 1, the units of each scale are different.

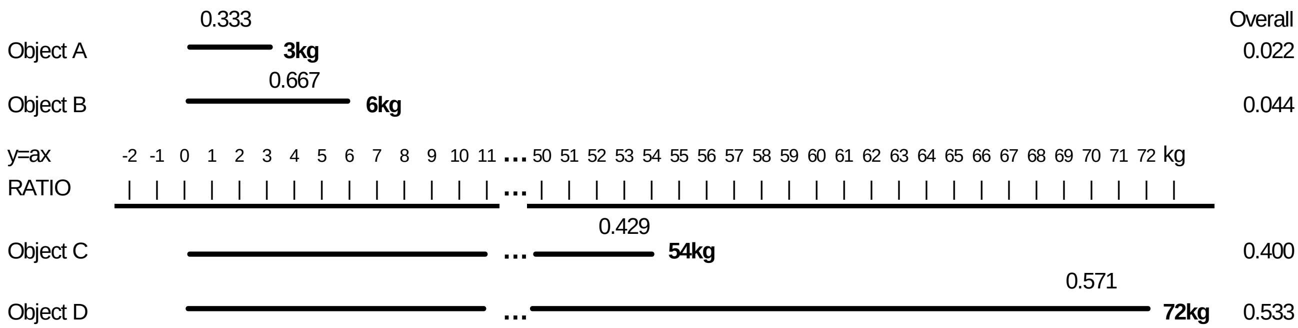

Figure 4: Two Relative Ratio Scales of Different Magnitudes

In order to combine the 4 objects into a common relative scale, it is necessary to relate the two scales across the different magnitudes. This can be done by referring to objects B and C that are within one order of magnitude: C/B = 9. With this information, we can generate the overall priorities shown at the right of Figure 4. These overall sum-to-one priorities have a unit of measure of a hypothetical object that represents the sum of all four objects.

In Figure 4, we have four ratio scales: the kilogram scale and the relative ratio scales for A and B, C and D, and all four objects. In all cases, the origin of the scales is natural zero. Because an object representing zero does not figure into the calculation of relative values, zero is usually left out of the scale. We should remember, however, that natural zero is the origin for all values on a ratio scale. As is done on the kilogram scale, it would be good practice to include zero as part of a relative ratio scale. It does not affect the scale, but it reminds us that the origin of measurement is from that value.

8. Observations, Comments and Conclusions

So, where is natural zero? That is the question posed in the title.

As depicted in Figures 3 and 4, it is the demarcation point between positive and negative values. In history, it took considerable time for this numerical position to be recognized. That is because zero is not something real – it and negative numbers were not amenable to counting. Although zero is a concept of nothingness, it is very much alive. We can have zero balances, zero slopes, zero gravity, and zero origin.

For the measurement of preferences, zero or no preference is a possible evaluation for some object, state or event. In other words, the object exists, but preference for it is neither positive nor negative. Because zero represents the absence of what is being measured, it is not directly involved in the determination of ratio scales. Since it is impossible to divide by zero, a ratio with zero as part of the calculation is not allowed. Instead, real objects with non-zero preference values are compared. In the AHP/ANP, it is the preference intensity for two different real objects that forms the units in paired comparisons. While all potential units of a ratio scale are measured from natural zero, it is the intensities from zero preference, not zero itself that determines the scale.

When direct numerical questioning is used for AHP/ANP, it is clear that a ratio question is being asked – How many times is the dominant object preferred to the less dominant object when both are used as units of measure. With other entry mechanisms, such as the verbal mode, it is less evident that a ratio question is being asked. In spite of this possible flaw, it is clear that AHP/ANP is attempting to achieve ratio measurement.

In paired comparisons, either item could be treated as the unit of measure. By convention, the less dominant item is selected as the unit (the denominator), the two units cancel each other out, and we are left with a dimensionless absolute value. As pointed out, the word dimensionless may be misleading, because an absolute scale has a counting unit of 1. As well, the absolute values in each column of the comparison matrix are expressed as the number of units of the object listed at the top of the column. By the Archimedean balance principle, paired comparisons do have a unit of measure.

A continuing debate is whether paired comparisons are measured from natural zero, the origin of a ratio scale. Barzilai and others say that without identifying natural zero, measurement cannot be made from that origin. This approach is more appropriate for determined scales that are permanent. Preference scales, on the other hand, are derived and transient. At different times, a decision maker can have different preferences for the same objects. It would be difficult to formulate a determined scale that is permanent. Nevertheless, it is also important for derived scales be measured from absolute zero if they are to be in ratio form (Haig et al., 1986).

Somewhat unfortunately, AHP/ANP fails to take explicit recognition of natural zero. Perhaps this is because zero is a demarcation point and origin that does not take a central role in the calculation of priorities. While intensity or dominance of one item or another is sensed from the origin, it is the comparison of unit departures from zero that determines paired comparisons. The fundamental scale used for paired comparisons measures the absolute or counting difference from absolute zero. That absolute zero applies to all paired comparisons and is not lost in the normalization process. The resulting relative priorities have a fixed origin of zero and unit of measure that can vary. Sometimes the unit is relative to the sum of the objects, sometimes to the best object (the ideal) and sometimes to a benchmark set. The unit can vary, but zero cannot.

In theory, comparison intensities have to be sensed from absolute zero. Whether or not that occurs in practice is open to question. Standard AHP/ANP methodology excludes zero from the fundamental and derived scales and fails to caution the DM that all comparison intensities have an origin of zero preference. By simply making zero more explicit, AHP/ANP methodology can be made more immune to errors.

Both Stevens (1951, p. 25) and von Neumann and Morgenstern (1953 p. 17) use the example of heat measurement to illustrate the evolution of measurement from ordinal sensations (colder, warmer), to interval measurements of quantity of heat (Fahrenheit, Celsius) to ratio measurement of temperature (Kelvin). Von Neumann and Morgenstern use that example to point out, *a priori*, that the ultimate shape of a theory of measurement cannot be forecasted. “The historical development of the theory of heat indicates that one must be extremely careful in making negative assertions about any concept with a claim to finality.” There may be questions and problems associated with AHP/ANP, but we must recognize that the methodology is still evolving. With questioning, scientific skepticism, and investigation, the evolution will lead to less controversy and more definitive answers.

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