THE RELIABILITY OF DATA IN PAIRWISE COMPARISON MATRICES

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Hong Kong, July 13th, 2018

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 E_1, \ldots, E_n - alternatives to be compared $\mathbf{A} = (a_{ij}) \land a_{ij} \in \mathbb{R}_+ \land i, j \in \{1, \ldots, n\}$ – pairwise comparisons matrix (PC-matrix). $a_{ij} \simeq \frac{E_i}{E_j}$

- A is *reciprocal* if for all $i, j \in \{1, \ldots, n\}$: $a_{ij} = \frac{1}{a_{ii}}$.
- A is *consistent* if for all $i, j, k \in \{1, \ldots, n\}$: $a_{ij} \cdot a_{jk} = m_{ik}$.
- For i < j < k each triple $\{a_{ij}, a_{jk}, a_{ik}\}$ is called a *triad*.

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A – a reciprocal square matrix The Eigenvalue Method (EV) (Saaty 1977): Find $\lambda > 0$ and $w = (w_1, \dots, w_n)$ with positive coordinates such that

 $\mathbf{A}\mathbf{w} = \lambda\mathbf{w}$

and $(\exists v = (v_1, \dots, v_n) \neq 0 \ \mathbf{A}v = \mu v) \Rightarrow |\mu| \le |\lambda|.$

Number λ is an eigenvalue. Vector w is called a principal right eigenvector and it serves as a priority vector.

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Consistency Index

Basing on this method, Saaty introduced the Consistency Index:

$$CI(\mathbf{A})=rac{\lambda-n}{n-1}.$$

REMARK

 $CI(\mathbf{A}) = \mathbf{0} \Leftrightarrow \mathbf{A}$ is consistent.

THEOREM (Aguarón and Moreno-Jiménez, 2003)

$$Cl(\mathbf{A}) = \frac{1}{n(n-1)} \sum_{i < j} e_{ij},$$

where

$$e_{ij}=a_{ij}rac{w_j}{w_i}+a_{ji}rac{w_i}{w_j}-2.$$

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Let $\mathbf{A} = [a_{ij}]$ be a pairwise comparison matrix. Fix a positive threshold ε .

The sketch of the algorithm of the input matrix improvement:

Repeat steps 1-2 until LCI(**A**) < ε . Step 1: Find *p*, *q* such that *p* < *q* and $e_{pq} = \max_{i < j} e_{ij}$. Step 2: Replace a_{pq} with $\frac{w_p}{w_q}$, and a_{qp} with $\frac{1}{a_{pq}}$.

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EXAMPLE 1

INPUT:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 0.25 & 1 & 1 \end{bmatrix}$$

and a threshold $\varepsilon = 0.01$.

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STEP 1:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 0.25 & 1 & 1 \end{bmatrix} \qquad \lambda_{MAX} = 3.217$$
$$\begin{bmatrix} w_i \\ w_j \end{bmatrix} = \begin{bmatrix} 1 & 1.5874 & 2.5198 \\ 0.6300 & 1 & 1.5874 \\ 0.3969 & 0.6300 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 2.51984 \\ 1.5874 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} e_{ij} \end{bmatrix} = \begin{bmatrix} 0 & 0.217361 & 0.217362 \\ * & 0 & 0.217361 \\ * & * & 0 \end{bmatrix} \qquad CI = 0.1085$$

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STEP 2: $\mathbf{A} = \left[\begin{array}{rrrr} 1 & 1 & 2.5198 \\ 1 & 1 & 1 \\ 0.3969 & 1 & 1 \end{array} \right]$ $\lambda_{MAX} = 3.09566$ $\begin{bmatrix} w_i \\ w_j \end{bmatrix} = \begin{bmatrix} 1 & 1.3616 & 1.8518 \\ 0.7344 & 1 & 1.3600 \\ 0.5400 & 0.7353 & 1 \end{bmatrix}$ $\begin{bmatrix} e_{ij} \end{bmatrix} = \begin{bmatrix} 0 & 0.096021 & 0.095657 \\ * & 0 & 0.95294 \\ * & * & 0 \end{bmatrix}$ $w = \begin{bmatrix} 1.85175\\ 1.36\\ 1 \end{bmatrix}$ *Cl* = 0.04783

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STEP 3:

$$\mathbf{A} = \begin{bmatrix} 1 & 1.3616 & 2.5198 \\ 0.7344 & 1 & 1 \\ 0.3969 & 1 & 1 \end{bmatrix} \qquad \lambda_{MAX} = 3.04225$$
$$\begin{bmatrix} w_i \\ w_j \end{bmatrix} = \begin{bmatrix} 1 & 1.6717 & 2.0524 \\ 0.5982 & 1 & 1.2278 \\ 0.4872 & 0.8145 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 2.0524 \\ 1.22775 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} e_{ij} \end{bmatrix} = \begin{bmatrix} 0 & 0.042246 & 0.0042250 \\ * & 0 & 0.042248 \\ * & * & 0 \end{bmatrix} \qquad CI = 0.021125$$

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STEP 4:

$$\mathbf{A} = \begin{bmatrix} 1 & 1.3616 & 2.0524 \\ 0.7344 & 1 & 1 \\ 0.4872 & 1 & 1 \end{bmatrix} \qquad \lambda_{MAX} = 3.01874 \\ \begin{bmatrix} w_i \\ w_j \end{bmatrix} = \begin{bmatrix} 1 & 1.5612 & 1.7900 \\ 0.6405 & 1 & 1.1466 \\ 0.5587 & 0.8722 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 1.79001 \\ 1.14658 \\ 1 \end{bmatrix} \\ \begin{bmatrix} e_{ij} \end{bmatrix} = \begin{bmatrix} 0 & 0.0187405 & 0.00187395 \\ * & 0 & 0.0187389 \\ * & * & 0 \end{bmatrix} \qquad CI = 0.00937 < \varepsilon$$

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EXAMPLE 2

INPUT:

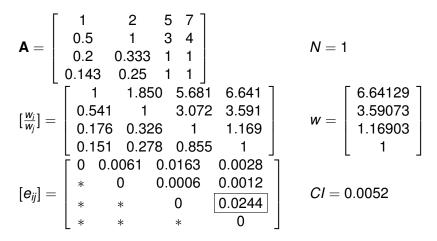
and a

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 0.5 & 1 & 3 & 4 \\ 0.2 & 0.33 & 1 & 1 \\ 0.143 & 0.25 & 1 & 1 \end{bmatrix}$$
threshold $\varepsilon = 0.001$.

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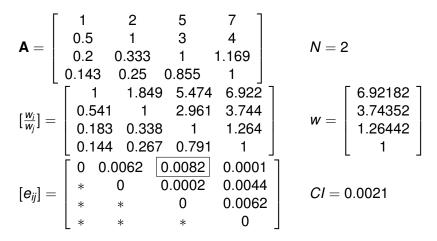
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STEP 1:



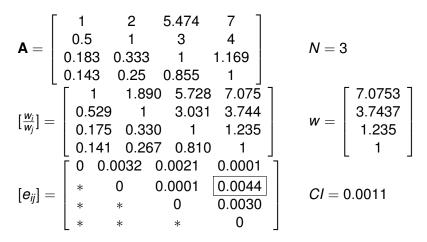
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STEP 2:



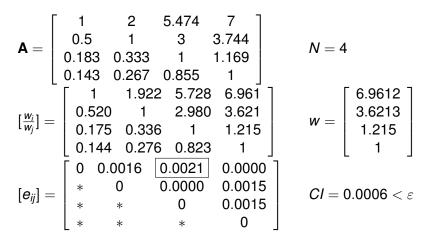
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STEP 3:



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STEP 4:



- simplicity;
- speed;
- preservation of most elements of the matrix;
- possibility of application in other methods (Least Squares Method, Logarithmic Least Squares Method).

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What is left to do?

• a formal proof of the algorithm correctness;

- more tests;
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THANK YOU FOR YOUR ATTENTION!

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