A χ^2 based approach to consistency evaluation of multiplicative preference relations

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Abstract

In this paper we develop the idea introduced by Lipovetski and Conklin in 2002 of considering a pairwise comparison matrix (PCM) as a contingency table. Instead of focusing on outliers detection, we use the χ^2 value as an index to evaluate the deviation of a pairwise comparison matrix from consistency. We verify by means of numerical simulations that our new index satisfies a set of five properties recently introduced in order to characterize an inconsistency index. Therefore, we argue that our χ^2 based index can be considered as a suitable index for evaluating the inconsistency of a PCM.

Keywords: analytic hierarchy process, χ^2, pairwise comparison matrices, consistency indices.

1 Introduction

In the Analytic Hierarchy Process (AHP) by T. Saaty [8], a pairwise comparison matrix (PCM) $\mathbf{A} = (a_{ij})$ is assumed to be a positive square reciprocal matrix, that is $a_{ij} > 0 \quad \forall i, j$; $a_{ij}a_{ji} = 1 \quad \forall i, j$. Conversely, the consistency condition,

$$a_{ik} = a_{ij}a_{jk} \quad \forall i, j, k \tag{1}$$

is, in general, not required. If (1) is satisfied, the matrix $\mathbf{A}$ is said to be *consistent* and, as a consequence, there exists a weight vector $\mathbf{w} = (w_1, \ldots, w_n)$ such that

$$a_{ij} = \frac{w_i}{w_j} \quad \forall i, j. \tag{2}$$

If (1) is violated, the matrix $\mathbf{A}$ is said to be *inconsistent*. Nevertheless, the problem of evaluating to which extent the condition (1) is violated is considered as a crucial point of the method and we refer to it as the problem of *consistency evaluation* of the elicited preferences. The best known method for consistency evaluation is the *consistency index, CI,* proposed by Saaty [8]. This index is based on the principal eigenvalue λ_{max} of the pairwise comparison matrix $\mathbf{A} = (a_{ij})$ and is defined as follows,

$$CI(\mathbf{A}) = \frac{\lambda_{max} - n}{n - 1}. \tag{3}$$

After that, several alternative indices have been introduced, most of them closely related to a particular prioritization method. All these indices aim to measure the global deviation of the elicited preferences a_{ij} from the estimated ratio of weights w_i/w_j [3]. Our proposal is to introduce a new point of view in consistency evaluation by considering the PCM $\mathbf{A} = (a_{ij})$ as a contingency table. This idea was introduced by Lipovetski and Conklin [7] in order to detect outliers in a PCM due to inaccurate data entry and random errors. In this paper we follow the same idea but we propose to consider the corresponding χ^2 value of $\mathbf{A} = (a_{ij})$ as an inconsistency index of the matrix. We justify our proposal by checking that this new index satisfies five properties recently introduced in order to characterize an inconsistency index [2].

2 A new Inconsistency Index

Our starting point is the observation that the entries w_{ij} of a consistent PCM can always be written in the form

$$w_{ij} = \frac{(\sum_{j=1}^{n} w_{ij})(\sum_{i=1}^{n} w_{ij})}{(\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij})}. \tag{4}$$

This was proved in [7] and follows from the characterization (2). By referring to the statistical point of view, such a matrix is called contingency table of two independent attributes: priority (in rows) and anti-priority (in columns). Note that independence of the two attributes corresponds to proportionality of the columns (rows), which is a characterizing property of a consistent PCM. We will refer to this type of matrix as to an 'expected' or 'theoretical' matrix. If we pass from this ideal matrix $\mathbf{W} = (w_{ij})$ to a PCM $\mathbf{A} = (a_{ij})$ which elements are the decision maker's numerical estimates of the preferences, then the matrix $\mathbf{A}$ is, in general, inconsistent and we will refer to this matrix as to an 'observed' or 'empirical' matrix. In this case, the expected values of the elements a_{ij} can be defined similarly to (4),

$$e_{ij} = \frac{(\sum_{j=1}^{n} a_{ij})(\sum_{i=1}^{n} a_{ij})}{(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij})}. \tag{5}$$

If the empirical (or observed) data a_{ij} are close to the expected (or theoretical) values e_{ij}, then matrix $\mathbf{A} = (a_{ij})$ is close to consistency. From the previous remarks, the χ^2 index is a suitable tool to evaluate the deviation of empirical data from the theoretical ones and we can give the following definition of our inconsistency index.

Definition 1 (χ^2 based inconsistency index)**.** *Given a pairwise comparison matrix* $\mathbf{A} = (a_{ij})$*, the* χ^2 *based inconsistency index* $I_{\chi^2}(\mathbf{A})$ *is defined as*

$$I_{\chi^2}(\mathbf{A}) = \chi^2(\mathbf{A}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(a_{ij} - e_{ij})^2}{e_{ij}} \ , \tag{6}$$

where e_{ij} *are the expected values given by (5).*

From definition (6), it is clear that the smaller $I_{\chi^2}(\mathbf{A})$ is, the more the data support consistency of $\mathbf{A}$. Nevertheless, in the next section we give a more formal validation of our definition.

3 Characterizing Properties

In order to justify the definition (6), we consider five axiomatic properties, denoted by P1, P2, P3, P4 and P5 which has been recently defined and discussed in [2] in order to characterize an inconsistency index. We avoid going into details and the interested reader can refer to [2] for a more precise description of the properties. In the following, we will prove that $I_{\chi^2}(\mathbf{A})$ satisfies properties P1, P2 and P5, whereas properties P3 and P4 have been verified by means of numerical simulations. Given a PCM $\mathbf{A}$, The first characterizing property (P1) requires that an inconsistency index $I(\mathbf{A})$ assumes a unique real number, say ν, if and only if the matrix $\mathbf{A}$ is consistent. From definition 1, it is straightforward that index $I_{\chi^2}(\mathbf{A})$ is null if and only if $\mathbf{A}$ is consistent, so that $I_{\chi^2}(\mathbf{A})$ satisfies P1 with $\nu = 0$. Property P2 requires that $I(\mathbf{A})$ should be independent from the order of the alternatives. It follows directly from (6) that $I_{\chi^2}(\mathbf{A})$ satisfies also P2. Given that the only transformation $a_{ij} \to f(a_{ij})$ which preserves the reciprocity property is $f(a_{ij}) = (a_{ij})^k$, the idea underlying property P3 is the following: if inconsistent preferences are intensified by choosing in $f(\cdot)$ an arbitrary $k > 1$, then a better value of an inconsistency index cannot be obtained. By 'preference intensification' we mean going farther from complete indifference $a_{ij} = 1 \ \ \forall i, j$, which is clearly fully consistent. Going farther from this uniformity means having stronger judgments and this should *not* make their possible inconsistency *less* evident. Numerical simulations confirm that P3 is always satisfied by the index $I_{\chi^2}(\mathbf{A})$. The property P4 refers to the modification of a single preference value in an *consistent* matrix. More precisely, the property requires that the larger is the change of the single preference from its consistent value, the more inconsistent will be the obtained matrix. Numerical simulations confirm that also P4 is always satisfied by the index $I_{\chi^2}(\mathbf{A})$.

The last property, (P5), requires the continuity of an inconsistency index. Continuity of $I_{\chi^2}(\mathbf{A})$ directly follows from continuity of the functions involved in its definition (6).

4 Concluding remarks

Some other known methods to derive weights from a pairwise comparison matrix and to evaluate its inconsistency refer to χ^2, but they are intrinsically different from our approach. We cite three of them which are based on the optimization of a deviation function, see [5] [9] [10], while a fourth method [6] proposes a consistency test based on the comparison between the Saaty's CI (3) and a threshold obtained by means of a maximum likelihood approach. The interested reader can find details in the references. We finally note that definition (6) involves no optimization problems. Future research will be focused in two directions. First, we will investigate the relationship / agreement between our index and the other most relevant inconsistency indices, following the approaches proposed in [1] and in [4]. Second, we will define suitable thresholds for our index in order to be able to correctly classify a PCM as acceptably consistent, mainly referring our proposals to the well known 0.1 threshold by Saaty.

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