AN APPLICATION OF INCOMPLETE PAIRWISE COMPARISON MATRICES FOR RANKING TOP TENNIS PLAYERS

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ABSTRACT

Pairwise comparison matrices (PCM) are important tools in multi-attribute decision making. Our paper presents a special application of PCMs: ranking of professional tennis players based on their results against each other. The selected 25 players have been on the top of the ATP rankings for a shorter or longer period in the last 40 years. Some of them have never met on the court. That property of the comparisons led us to a special class of PCMs: to the application of incomplete pairwise comparison matrices. One of the aims of the paper is to provide ranking of the selected players, however, the analysis of incomplete pairwise comparison matrices is also in the focus.

Keywords: incomplete pairwise comparison matrix, ranking.

1. Introduction

Our aim is to make a "historical" comparison of the best tennis players of the last 40 years. The ranking idea is how the players performed against each other in a pairwise manner in the long run. There is a freely available database about the results of the top tennis players including data from 1973. That gave the possibility to construct the pairwise comparison matrices of those players who have been leading the ATP ranking for a period of any length. Applying one of the estimation methods for generating a weight vector we can produce an order of the players: a ranking.

2. Literature Review

In recent years some papers have attempted to rank professional tennis players with several methods. Radicchi [2011] considered all matches played between 1968 and 2010 to construct a weighted and directed preference graph. The author developed a diffusion algorithm similar to Google's PageRank to derive the ranking of nodes representing the tennis players. Dingle et al. [2013] used PageRank-based tennis rankings instead of the official ATP and WTA rankings. Motegi and Masuda [2012] proposed a network-based dynamical ranking system taking into account that the strength of a player depends on time.

Our approach differs from the above-mentioned applications in its methodological background, using incomplete pairwise comparison matrices. One of the authors applied the PCM-based methodology previously for Swiss-type chess tournaments [Csató, 2013].

3. Hypotheses/Objectives

We have to emphasize that we do not want to replace the ATP-ranking with another methodology. One of the major goals of the ATP-ranking is to give help to the organizers in the seeding procedure to avoid those situations when either the "best" players meet each other during the first rounds or relatively weak players can walk to the semifinals or final. For this purpose the recent ATP-ranking of the players is useful, because short-term view is needed, and the results of the last season can provide a solid basis. In our "ranking" we use results of a long period. Contrary to the ATP ranking where winning is important, but the person who was beaten is not relevant; and the calculation does not use concrete scores, but the final results of the tournaments, we will use face to face match results.

4. Research Design/Methodology

In our case the alternatives are tennis players. Choosing any two of them (A_i and A_j) we can tell the results of all matches have been played against each other. Let the number of winning matches of A_i over A_j be x, and the number of lost matches y. We can construct the ratio x/y: if the ratio is greater than 1, we will say that A_i is a „better" player than A_j. In case of x/y is equal to 1 we are not able to decide who is the better. Let the a_{ij} element of the pairwise comparison matrix $\boldsymbol{A}$ be x/y, and the a_{ji} element be y/x.

The PCM matrix $\boldsymbol{A}$ is used to determine a weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, $w_i > 0$ ($i = 1, \ldots, n$) where the a_{ij} elements are estimated by w_i/w_j with the best fitness. Since the estimated values are ratios, it is a usual normalization condition that the sum of the weights is equal to 1: $\Sigma\, w_i = 1$. That estimation problem can be formulated in several ways. Saaty [1980] formulated the eigenvalue problem in the Analytic Hierarchy Process (AHP), where the components of the right eigenvector belonging to the maximal eigenvalue (λ_{max}) of matrix $\boldsymbol{A}$ will give the weights. We will refer to that method as the Eigenvector Method (EM). For solving the estimation problem it could be obvious to apply methods based on distance minimization, too. That approach will estimate the elements of the $\boldsymbol{A}$ matrix with the elements of a matrix $\boldsymbol{W}$, where the w_{ij} element of $\boldsymbol{W}$ is w_i / w_j, w_i and $w_j > 0$, ($i = 1, \ldots, n; j = 1, \ldots, n$) and the objective function to be minimized is the distance of the two matrices, for instance in the sense of logarithmic least squares (LLSM; Crawford and Williams, 1985).

An incomplete PCM differs from the complete PCM having some unknown elements (Harker, 1987). There could be several reasons why the elements of a PCM are missing. It can happen that the decision makers do not have time to make all comparisons, or they are not able to make some of the comparisons. Some data could have lost, but it is also possible that the comparison was not possible. In our case the reason of missing elements is absolutely obvious: we are not able to compare those players directly who have never played against each other.

Applying the EM and the LLSM methods to the optimal completion of the incomplete matrices we can make our rankings. The optimal completion has been worked out by Bozóki et al. [2010]. The idea is that those elements will be accepted which make the maximal eigenvalue of the matrix minimal. It can be shown that this optimization problem has a unique solution if and only if the graph of the PCM is connected. It can be found in Bozóki et al. [2010], too, that the same condition is necessary for the linear equation system derived from the LLSM problem.

5. Data/Model Analysis

We used data from the ATP website. The players who have been included in our ranking had overlapping professional carrier periods. Therefore the associated graph was connected and the method of Bozóki et al could be used for the optimal completion. We have made some minor transformation on the data to avoid problems with zero wins or losses, or biases caused by players who have played very few matches against each other.

The final rankings produced with the two methods are not far from each other. Nadal, Federer and Sampras are the top three in all calculations. Lendl, Borg, Becker, Djokovic and Agassi are the followers. Tennis fans can debate the rankings, of course; however, the methodology was proved to be applicable. Variations in the rankings and their possible reasons will be discussed in the presentation.

6. Conclusions

Some sports create a natural environment for using incomplete pairwise comparisons to obtain a ranking of either players or teams. Our application provides an example to analyze some properties of the matrices and the consequences for the final results. What is the impact of the density of the matrix, or the distribution of the degree of vertices? In sport competitions it happens frequently that A beats B, B beats C, and C beats A. The ratio of non-transitive triads compared to the number of all triads indicates a higher or lower degree of the strongest case of inconsistency: intransitivity. Can we measure inconsistency with the help of non-transitive triads (Kéri, 2011)? Several questions are open for further research.

7. Key References

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