GROUP DECISION AS APPROXIMATION OF INDIVIDUAL INTERVAL WEIGHTS BY INTERVAL AHP

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ABSTRACT

The individual decision in this study is denoted as interval weights of alternatives. Based on the idea that the inconsistency among comparisons stems from the uncertainty of the weights in a decision maker's mind in giving them, the uncertain weight is assumed as interval in Interval AHP. Then, the group interval weight is obtained as an approximation of the individual interval weights based on the satisfaction and dissatisfaction of each decision maker. The condition of the group decision is to have some common to all decision makers' for a consensus. S/he is satisfied more with the group decision, as it reflects his/her decision more. The satisfaction is defined as the range of the group decision supported by him/her. While, s/he is dissatisfied with the group decision, when it is different from his/hers so that the dissatisfaction is defined as the range of the group decision which is not supported by him/her. In the proposed model, the satisfaction and dissatisfaction is maximized and minimized, respectively, under the group decision condition. As a result the deviations of the upper and lower bounds of the group and individual interval weights are minimized.

Keywords: group decision making, interval analysis, analytic hierarchy process

1 Introduction

In AHP, the priority weights of alternatives are obtained as real values from the pairwise comparison matrix whose elements are real values and given by a decision maker (Saaty, 1980). The well-known techniques are geometric mean and eigenvector methods. In order to reflect the inconsistency among the given comparisons, the uncertain weights are assumed as interval in Interval AHP (Sugihara & Tanaka, 2001). The interval weights are obtained so as to include the given comparisons as close as possible. The group decision making is discussed from the viewpoint of AHP (Dyer & Forman, 1992; Basal & Saaty, 1993). There are two ways to aggregate the individuals into a group. One is to aggregate the individual judgments beforehand and the group decision is obtained from the aggregated judgments. The other is to aggregate the individual decisions independently obtained from the given judgments. In other words, the group decision is the approximations of the individual decisions (Entani & Inuiguchi, 2010). Since the latter aggregation shows a decision maker his/her decision, it helps him/her to understand the relation between his/her and the group decisions. This study follows the aggregation of individual decisions considering his/her satisfaction and dissatisfaction. They are defined based on the deviations of the group decision from the individual ones and in order to minimize them they are maximized and minimized, respectively.

2 Interval AHP

In AHP, decision maker k gives the pairwise comparison matrix $A_k = [a_{kij}]$, where a_{kij} is his/her intuitive judgment on the importance ratio of alternative i to that of alternative j so that $a_{kii} = 1$ and $a_{kij} = 1/a_{kji}$. The comparisons are consistent if and only if $a_{kij} = a_{kil}a_{klj} \forall i, j$. In Interval AHP, it is assumed that the given comparisons are inconsistent since an alternative is compared to the others $n-1$ times and its $n-1$ weights may not be always equal. Then, the weight of an alternative is assumed as an interval (Sugihara & Tanaka, 2001). The problem to obtain the interval weights $W_{ki} = [\underline{w}_{ki}, \overline{w}_{ki}]$ is formulated as the following LP problem.

$$\begin{array}{rl} \min & \sum_i (\overline{w}_{ki} - \underline{w}_{ki}), \\ \text{s.t.} & \sum_{i \neq j} \overline{w}_{ki} + \underline{w}_{kj} \geq 1,\ \sum_{i \neq j} \underline{w}_{ki} + \overline{w}_{kj} \leq 1\ \forall j, \\ & \dfrac{\underline{w}_{ki}}{\overline{w}_{kj}} \leq a_{kij} \leq \dfrac{\overline{w}_{ki}}{\underline{w}_{kj}}\ \forall (i, j), \\ & \underline{w}_{ki} \geq \epsilon\ \forall i, \end{array} \tag{1}$$

where the first constraints are for the normalization of intervals based on interval probability and the next inequalities require the interval weights to include the given comparisons. By minimizing the widths of the interval weights, they are as close as possible to the comparison because of the inclusion constraints. If the comparisons are consistent, the weights are obtained as real values $\overline{w}_{ki} = \underline{w}_{ki} \forall i$ and equal to those by geometric mean and eigenvector methods.

3 Group decision

In case of a group of m decision makers, there are m sets of individual interval weights are independently obtained from their comparison matrices by (1). Then, the group decision $W_i = [\underline{w}_i, \overline{w}_i]$ is considered to be their approximation and its condition is denoted as follows.

$$\underline{w}_{ki} \leq \overline{w}_i,\ \underline{w}_i \leq \overline{w}_{ki}\ \forall k \leftrightarrow \max_k \underline{w}_{ki} \leq \overline{w}_i,\ \underline{w}_i \leq \min_k \overline{w}_{ki}, \tag{2}$$

where the group decision should have some common to all the individual ones. Since the group decision satisfies each decision maker more as wider the range of their core becomes, the core measures his/her satisfaction. While, when s/he cannot support some of the group decision, its difference range from his/her decision measures his/her dissatisfaction. Figure 1 shows the examples of his/her satisfactions and dissatisfactions of an alternative. An individual interval weight is illustrated as the top line and the following four lines are the possible group ones which satisfy (2). The satisfaction by decision maker k, denoted as α_{ki}, is defined as follows.

$$\alpha_{ki} = \min\{(\overline{w}_i - \underline{w}_{ki}), (\overline{w}_{ki} - \underline{w}_i), (\overline{w}_i - \underline{w}_i), (\overline{w}_{ki} - \underline{w}_{ki})\}, \tag{3}$$

which represents how much decision maker k supports the group decision so that it should be maximized as $\max \sum_{ki} \alpha_{ki}$.

As for the dissatisfaction, denoted as β_{ki}, is defined as follows.

$$\beta_{ki} = \max\{(\underline{w}_{ki} - \underline{w}_i), (\overline{w}_i - \overline{w}_{ki}), 0, (\underline{w}_{ki} - \underline{w}_i + \overline{w}_i - \overline{w}_{ki})\} \tag{4}$$

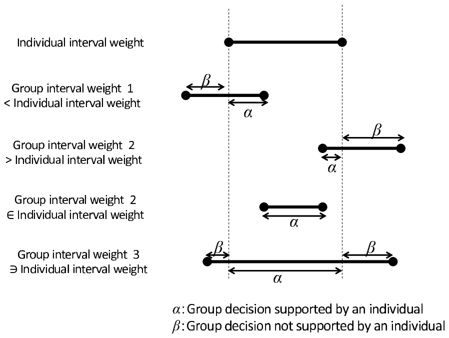

Figure 1: Individual and group decisions as interval weights

which represents how much decision maker k does not support the group decision so that it should be minimized as min $\sum_{ki} \beta_{ki}$.

As shown in Fig. 1, the group decision is divided into the satisfaction and dissatisfaction as $\overline{w}_i - \underline{w}_i = \alpha_{ki} + \beta_{ki}$. From the viewpoint of comparing the upper and lower bounds of the individual and group interval weights, the group decision is obtained as the approximations of the individual decisions by maximizing and minimizing the satisfaction and dissatisfaction. The problem is formulated as follows.

$$
\begin{array}{ll}
& \max\ \sum_{ki} \alpha_{ki},\ \min\ \sum_{ki} \beta_{ki}, \\
\text{s.t.} & \underline{w}_{ki} \leq \overline{w}_i,\ \underline{w}_i \leq \overline{w}_{ki}\ \forall i, k, \\
& \overline{w}_i - \underline{w}_{ki} \geq \alpha_{ki},\ \overline{w}_{ki} - \underline{w}_i \geq \alpha_{ki},\ \overline{w}_i - \underline{w}_i \geq \alpha_{ki},\ \overline{w}_{ki} - \underline{w}_{ki} \geq \alpha_{ki}\ \forall i, k, \\
& \underline{w}_{ki} - \underline{w}_i \leq \beta_{ki},\ \overline{w}_i - \overline{w}_{ki} \leq \beta_{ki},\ 0 \leq \beta_{ki},\ \underline{w}_{ki} - \underline{w}_i + \overline{w}_i - \overline{w}_{ki} \leq \beta_{ki}\ \forall i, k. \\
& \sum_{i \neq j} \overline{w}_i + \underline{w}_j \geq 1,\ \sum_{i \neq j} \underline{w}_i + \overline{w}_j \leq 1\ \forall j, \\
& \epsilon \leq \underline{w}_i \leq \overline{w}_i\ \forall i,
\end{array}
\tag{5}
$$

where the variables the upper and lower bounds of the group interval weight, $\underline{w}_i, \overline{w}_i$, and the individual satisfaction and dissatisfaction, α_{ki}, β_{ki}.

When the group decision includes the individual decision as $W_{ki} \subseteq W_i$, decision maker k is fully satisfied with it and his/her satisfaction equals to the width of his/her interval weight $\alpha_{ki} = \overline{w}_i - \overline{w}_{ki}$. Focusing on maximizing the satisfaction, the group decision is obtained so as to include all the individual decisions as $W_i = [\min_k \underline{w}_{ki}, \max_k \overline{w}_{ki}]$. As a result, the width of the group interval weight tends to be wide. While, focusing on minimizing the dissatisfaction, when the group decision is included in the individual decision as $W_i \subseteq W_{ki}$, decision maker k is never dissatisfied with it $\beta_{ki} = 0$. In such a case, the group decision is $W_i = [\max_k \underline{w}_{ki}, \min_k \overline{w}_{ki}]$, if $\max_k \underline{w}_{ki} \leq \min_k \overline{w}_{ik}$, i.e., all the individual decisions have some common each other. This condition is striker than the condition of the group decision (2) which does not require the relations among the individual decisions. It seldom happens that there are intersections of all the individual decisions so that the minimum of the dissatisfaction is usually more than 0, $\sum_{ki} \beta_{ki} > 0$.

For calculation, two objective functions are aggregated by the weighting approach as max $\lambda \sum_{ki} \alpha_{ki} - (1-\lambda) \sum_{ki} \beta_{ki}$, where λ and $(1-\lambda)$ are the weights for satisfaction and dissatisfaction, respectively. The individual decision as interval weights of alternatives by (1) reflects the inconsistency among the given comparisons. For a decision maker, since all the possibilities in his/her judgments have been considered into his/her decision, the group decision which cannot be supported by such a possible individual decision may be unacceptable. From this viewpoint, it is reasonable to primely minimize the dissatisfaction and secondarily maximize the satisfaction as $1 - \lambda > \lambda \leftrightarrow 0.5 > \lambda$.

As a decision maker gives more inconsistent comparisons, the widths of his/her interval weights become wider. Because of the wide width, his/her dissatisfaction tends to be small. In case of completely consistent comparisons, his/her satisfaction always equals to 0. In this way, the individual satisfaction and dissatisfaction depend on the inconsistency of the given comparisons. It is difficult to control them by assuming their thresholds. Instead, the most and the least satisfaction and dissatisfaction among all the decision makers are limited for the fairness of the decision makers.

$$\begin{array}{rl} \min & \lambda(\overline{\alpha} - \underline{\alpha}) + (1-\lambda)(\overline{\beta} - \underline{\beta}) \\ s.t. & \underline{\beta} \leq \sum_i \beta_{ki} \leq \overline{\beta},\ \underline{\alpha} \leq \sum_i \alpha_{ki} \leq \overline{\alpha}\ \forall k, \end{array} \tag{6}$$

4 Numerical example

At first, three decision makers give the comparison matrices of 4 alternatives independently.

$$A_1 = \begin{bmatrix} 1 & 2 & 4 & 8 \\ - & 1 & 2 & 4 \\ - & - & 1 & 2 \\ - & - & - & 1 \end{bmatrix} A_2 = \begin{bmatrix} 1 & 3 & 3 & 4 \\ - & 1 & 3 & 3 \\ - & - & 1 & 4 \\ - & - & - & 1 \end{bmatrix} A_3 = \begin{bmatrix} 1 & 1 & 4 & 6 \\ - & 1 & 1 & 2 \\ - & - & 1 & 3 \\ - & - & - & 1 \end{bmatrix}$$

Then, by (1), each decision is obtained as a set of interval weights of alternatives.

$$W_1 = \begin{bmatrix} 0.533 \\ 0.267 \\ 0.133 \\ 0.067 \end{bmatrix} W_2 = \begin{bmatrix} 0.571 \\ [0.190, 0.214] \\ [0.071, 0.190] \\ [0.048, 0.143] \end{bmatrix} W_3 = \begin{bmatrix} 0.390 \\ [0.244, 0.309] \\ [0.098, 0.244] \\ [0.081, 0.122] \end{bmatrix}$$

Because of completely consistent comparisons by decision maker 1, A_1 his/her decision, W_1 is denoted as real values. They equal to those by geometric mean and eigenvector methods. Since the individual decision reflect the possibilities in the given comparisons, it is reasonable for the weight of the dissatisfaction $1 - \lambda$ to be more than that of satisfaction λ. As for $1 - \lambda > \lambda$, the group decisions with $\lambda = 0.1$ and 0.4, and in addition that with $\lambda = 0.9$ are shown.

$$W(\lambda = 0.1) = \begin{bmatrix} [0.390, 0.571] \\ [0.214, 0.267] \\ [0.133, 0.134] \\ [0.067, 0.081] \end{bmatrix} W(\lambda = 0.4) = \begin{bmatrix} [0.390, 0.571] \\ [0.214, 0.267] \\ [0.098, 0.190] \\ [0.067, 0.122] \end{bmatrix}$$

$$W(\lambda = 0.9) = \begin{bmatrix} [0.390, 0.571] \\ [0.190, 0.390] \\ [0.071, 0.244] \\ [0.048, 0.143] \end{bmatrix}$$

As for alternative 1, whose individual weights by all decision makers are real values, the satisfaction cannot be more than 0 and the dissatisfaction is the width of the group interval weight. Then, its group interval weight includes three individual weights with minimum width so that it is from their minimum to their maximum as $W_1 = [\min\{w_{11}, w_{21}, w_{31}\}, \max\{w_{11}, w_{21}, w_{31}\}]$. In case of $\lambda = 0.9$, where the satisfaction is primarily maximized, all the individual interval weights are included in the group ones as $W_{ki} \subseteq W_i$.

The group interval weights with λs are not very different, since λ only controls the weights for the satisfaction and dissatisfaction. The smaller λ becomes, the smaller the width of the group interval weight is. The group interval weight with small width decreases dissatisfaction, as well as satisfaction. The satisfaction α_k and dissatisfaction β_k of decision maker k with $\lambda = 0.1$ and 0.4 are compared.

$$\begin{aligned} \lambda = 0.1: \quad & (\alpha_1, \alpha_2, \alpha_3) = (0, 0.015, 0.024),\ (\beta_1, \beta_2, \beta_3) = (0.249, 0.234, 0.225), \\ \lambda = 0.4: \quad & (\alpha_1, \alpha_2, \alpha_3) = (0, 0.147, 0.156),\ (\beta_1, \beta_2, \beta_3) = (0.381, 0.234, 0.225). \end{aligned}$$

As the increase of the weight for satisfaction λ, the satisfactions of decision makers 2 and 3 increase and correspondingly the dissatisfaction of decision maker 1 increases.

5 Conclusion

The individual and group decisions in this study are denoted as interval weights of alternatives. The group decision is obtained so as to have some common to all individual decisions and consists of the satisfaction which is supported by a decision maker and the dissatisfaction which is not supported. The group interval weight is the approximation of the individual ones and their deviations of the upper and lower bounds are minimized by maximizing satisfaction and minimizing dissatisfaction, respectively.

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