A Grey Number Approach to Establish Judgment Matrix in AHP

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Abstract

Multiple attribute decision making (MADM) is becoming an important part of modern decision science. It has been extensively applied to various areas such as society, economics, management, etc., and has been receiving more and more attention over the last decades. However, owing to the increasing complexity of decision, the uncertainty of decision information growing sharply and the multi-period multi-attribute decision making has become the focus of people's attention. Therefore, this study proposes a multiple attribute decision making model (MADM) which takes the AHP technique as main structure, integrating the concepts of grey number into it to cope with uncertain information. An emerging market stock selection example is employed to demonstrate the feasibility and practicability of the proposed model. Results show that the proposed model is efficient and robust, and is practical for real world applications.

Keywords: *Comparison matrix; AHP; Multi-attribute decision making; Grey number*

1. Introduction

Multi-attribute decision making (MADA) problem is to select an appropriate alternative (choice) from a finite number of feasible alternatives based on the features of each attribute (objective or criteria) with respect to every alternative. AHP （technique for organizing and analyzing complex decisions）, proposed by Saaty (1977, 1980, 1986) is one of the most widely used techniques to solve MADM problems. AHP provides a comprehensive and rational framework for structuring a decision problem, for representing and quantifying its elements, for relating those elements to overall goals, and for evaluating alternative solutions. The basic concept of AHP technique is that AHP is a decision analysis technique used to evaluate complex multi-attributed alternatives with conflicting objectives among one or more actors. The process involves hierarchical decomposition of the overall evaluation problem into sub-problems that can be easily understood and evaluated. In

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recent years, AHP has been successfully adopted to solve various MADM problems, such as accounting (Webber et al. 1996), assessment (Sloane et al. 2003), programming (Ngai, 2003;Yang and Kuo, 2003), and information management (Liu & Shih, 2006). Hence, we use the AHP to resolve the MADM problem and extend it to the uncertain environment on account of its effectiveness and practicability.

A problem in applying the AHP is that the judgments are not transitive, i.e. the judgment matrix is inconsistent. For this reason, this paper starting from the objective property of grey number division operation, modify the definition of judgment matrix and make it only has reciprocity. Then, we can give out the determine condition of consistency for judgment matrix to overcome this problem and also to enhance the reliability of decision.

Besides, along with the increasing complexity of decision, the uncertainty of evaluation will be growing. Under this situation, the decision making scheme is not enough to reflect the decision environment, or the understanding of the expert is far from being comprehensive and accurate, at this time, the judgment has a variety of possible and decision maker(s) are unable to point out a precise numbers to express the important degree in paired comparisons. But, if they can give approximate ranges, i.e., grey numbers or interval numbers, by their knowledge and cognition. This way will represent their judgment and feelings more truly. Therefore, applying the concepts and operations of grey number will be helpful to deal with the uncertain information.

Hence, this study attempts to propose an effective MADM model, which adopts AHP as the main structure as well as integrating with the concepts of grey number, to effectively deal with the uncertain information and comprehensively aggregate the different decisions among all periods.

This paper is organized as follows: Section 2 describes the concepts of AHP technique and grey number, respectively. Based on the concepts in Section 2, Section 3 discusses the properties of consistent judgment matrix and presents the criteria of consistency in grey numbers. Finally, Section 4 concludes the paper.

2. Literature survey

To arrange the survey in various aspects, we divide it into two parts: the AHP technique and the concept of grey number.

2.1 AHP technique

Analytic hierarchy process (AHP, Saaty, 1977, 1980, 1990) is a multi-criteria decision making method based on pairwise comparisons for elements in a hierarchy. It decomposes problems in a hierarchical structure, and explicitly incorporates decision makers' expertise/experience in AHP evaluation. Decision makers make use of the subjective judgments, but can also integrate objectively measured information when necessary. The three principles of AHP are decomposition, comparative judgment, and synthesis.

In the above-mentioned procedure of AHP technique, a problem can be obviously identified. This problem can be further observed from Eq. (1) and (2). After obtaining the judgment matrix, the diagonal elements is defined as a fixed value 1, this man-made regulation led to a contradiction of judgment matrix itself. The reason is, in social and economic systems, decision makers' subjective judgment, choices and preference has a great influence on decision results. It is difficult to obtain the priorities of the alternatives or the weight of every criterion directly, even is impossible. However, this problem can be overcome by using the grey numbers (Deng, 1982).

2.2 Grey number

Grey system theory proposed by Deng (1982), is a mathematical theory originating from the concept of grey set. It can effectively solve uncertainty problems under discrete data and incomplete information. In grey system, if the system information is fully known, the system is called a white system. When the system information is unknown, it is called a black system. A system with partial information known and partial information unknown is grey system. Thus, a grey system contains uncertain information presented by grey number and grey variables, as shown in Figure 2.

Figure 2. The concept of grey system.

Let $\otimes a = [\underline{a}, \overline{a}] = \{a \mid \underline{a} \le a \le \overline{a}; \underline{a}, \overline{a} \in R\}$. Then, $\otimes a$ have two real numbers $\underline{a}$ (the lower limit of $\otimes a$) and $\overline{a}$ (the upper limit of $\otimes a$) is defined as follows (Liu & Lin, 2006)

- If $\underline{a} \to -\infty$ and $\overline{a} \to +\infty$, then $\otimes a$ is called the black number which means without any meaningful information or information is totally unknown.
- Else if $\underline{a} = \overline{a}$, then $\otimes a$ is called the white number which means with complete information or information is totally known.
- Otherwise, $\otimes a = [\underline{a}, \overline{a}]$ is called the grey number which means insufficient and uncertain information.

Grey number is a concept of grey theory to deal with the insufficient and incomplete information. Although grey theory has been applied in various fields (Liu & Shih, 2006), the applications are mostly based on the white numbers. Nevertheless, the obtained information from real world is always uncertain or incomplete. Hence, extending the applications from white number to grey number is necessary for real world applications.

3. The proposed multi-attribute decision making model

3.1 Grey number approach to establish the judgment matrix in AHP

We first discuss the grey number judgment matrix.

Definition 3.1. Let $D(\otimes)=[\otimes_{ij}]_{n\times n}$ be a grey number matrix and $i,j=1,2,\cdots,n$ such that

- $\otimes_{ij}=[\underline{d}_{ij},\overline{d}_{ij}]$,and $\frac{1}{9}\le \underline{d}_{ij}\le \overline{d}_{ij}\le 9$
- $\otimes_{ij}=\frac{1}{\otimes_{ji}}$

Then, $D(\otimes)$ is called grey number judgment matrix.

Suppose $D(\otimes)=[\otimes_{ij}]_{n\times n}$ is a grey number judgment matrix, $w(\otimes)=(w_1(\otimes),w_2(\otimes),\cdots,w_n(\otimes))^T$ is the grey number weight vector associated $D(\otimes)$, then, we have $\otimes_{ij}=\frac{w_i(\otimes)}{w_j(\otimes)}$ available for all $i,j=1,2,\cdots,n$.

Definition 3.2. Suppose $D(\otimes)=[\otimes_{ij}]_{n\times n}$ is a grey number judgment matrix and $i,j,k=1,2,\cdots,n$ be such that

$$\otimes_{ij}=\frac{1}{\otimes_{ji}},\quad \otimes_{ij}\times\otimes_{jk}=\otimes_{ji}\times\otimes_{ik} \tag{1}$$

Then, $D(\otimes)$ is called consistent grey number judgment matrix, and Eq. (1) is the consistency condition of $D(\otimes)$.

Next, we discuss the properties of consistent grey number judgment matrix. We first introduce the quasi-uniformity concept of real number matrix.

Theorem 3.1. A necessary and sufficient condition for $D(\otimes)=[\otimes_{ij}]_{n\times n}$ is consistent grey number matrix is there exist grey number $\otimes_i(i=i,2,\cdots,n)$, such that

$$\otimes_{ij}=\frac{\otimes_i}{\otimes_j},(i,j=1,2,\cdots,n) \tag{2}$$

3.2. The modeling mechanism of the grey number multi-attribute decision making model

Base on the operations of grey number, this paper proposes an effective multi-attribute decision making model under the condition of uncertain information. Before we describe the detailed model, we assume a positive grey number judgment matrix, $D(\otimes)=[\otimes_{ij}]_{n\times n}$.

$$D(\otimes)=[\otimes_{ij}]_{n\times n}=\begin{bmatrix}\otimes_{11} & \otimes_{12} & \cdots & \otimes_{1n}\\ \otimes_{21} & \otimes_{22} & \cdots & \otimes_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \otimes_{n1} & \otimes_{n2} & \cdots & \otimes_{nn}\end{bmatrix} \tag{3}$$

where $\otimes_{ij}$ denotes the grey number evaluations of the ith alternative with respect to the jth attribute. $\otimes_i=(\otimes_{i1},\otimes_{i2},\cdots,\otimes_{in})$ is the grey number evaluation series of the ith alternative. Assume $w(\otimes)=(w_1(\otimes),w_2(\otimes),\cdots,w_n(\otimes))^T$ is the grey number eigenvector of $D(\otimes)$, and the entries of grey number judgment matrix $D(\otimes)$ is changed at different decision period.

In order to ensure the scientificalness and correction of the proposed MADM model, we first introduce the modeling mechanism.

Theorem 3.2. Suppose $D(\otimes)=[\otimes_{ij}]_{n\times n}$ is consistent grey number matrix, $\underline{w}=(\underline{w}_1,\underline{w}_2,\cdots,\underline{w}_n)^T$ and $\overline{w}=(\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_n)^T$ are nonnegative normalized eigenvector of $(\underline{d}_{ij})_{n\times n}$ and $(\overline{d}_{ij})_{n\times n}$ associated the maximal eigenvalue. Then

$$w(\otimes)=[p\underline{w},q\overline{w}]=(w_1(\otimes),w_2(\otimes),\cdots,w_n(\otimes))^T \tag{4}$$

and a necessary and sufficient condition for $\otimes_{ij}=\frac{w_i(\otimes)}{w_j(\otimes)},(i,j=1,2,\cdots,n)$ is

$$\Omega=\sum_{j=1}^{n}\quad=\frac{1}{\sum_{i=1}^{n}\overline{d}_{ij}}=\frac{1}{\sum_{j=1}^{n}\frac{1}{\sum_{i=1}^{n}\underline{d}_{ij}}} \tag{5}$$

Next, the procedure of the proposed model can be shown as the following five steps.

Step 1: Construct the grey number judgment matrices.

Step 2: Calculate the maximal eigenvalue of grey number matrix $(\underline{d}_{ij})_{n\times n}$ and $(\overline{d}_{ij})_{n\times n}$ and the associated nonnegative normalized eigenvector

$\underline{w}=(\underline{w}_1,\underline{w}_2,\ldots,\underline{w}_n)^T$ and $\overline{w}=(\overline{w}_1,\overline{w}_2,\ldots,\overline{w}_n)^T$.

Step 3: Determine parameter p and q.

$$p=\sqrt{\sum_{j=1}^{n}\frac{1}{\sum_{i=1}^{n}\overline{d}_{ij}}},\quad q=\sqrt{\sum_{j=1}^{n}\frac{1}{\sum_{i=1}^{n}\underline{d}_{ij}}}$$

Step 4: Normalize the weight vector.

$$w(\otimes)=[p\underline{w},q\overline{w}]=(w_1(\otimes),w_2(\otimes),\ldots,w_n(\otimes))^T$$

Step 5: Rank the preference order.

A set of alternatives now can be preference ranked by the descending order of the weight vector.

4. Conclusions

Due to the increasing complexity of decision, the uncertainty in evaluation will be growing. It is difficult for decision makers to make evaluations with precise number. However, they can still use an approximate range of evaluation by their knowledge and cognition. Under this circumstance, dealing with the uncertain information is necessary on developing the decision making model.

In this study, we proposed a new grey number approach to modify the definition of judgment matrix in the original AHP technique, overcome the problem of inconsistency when the judgments are not transitive. Finally, we have integrated the above mentioned concepts, AHP technique to establish an effective multi-attribute decision making model.

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