On Properties of Pareto Optimal Weights from Pairwise Comparison Matrices Based on Graph Theory

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Pairwise comparisons

Pairwise comparison value a_{ij} is approximate of the ratio of weights between i -th alternative and j -th alternative:

$$a_{ij} \approx w_i / w_j$$

They are reciprocal symmetric.

$$a_{ij} = 1/a_{ji}, \qquad a_{ii} = 1$$

They are arranged into the pairwise comparison matrix $A = (a_{ij})$.

Inferring weights

Weights of alternatives are inferred from the values a_{ij}.

- In the completely consistent cases, $a_{ij} = w_i/w_j$.

The eigenvector method and the geometric mean method are widely used.

Multi-objective optimization

- The value a_{ij} is the approximate ratio of the weights.

$$a_{ij} \approx w_i / w_j$$

- Seeking weights can be regarded to as multi-objective optimization, such that, to find weights $\boldsymbol{w} = [w_1, \dots, w_n]$, where

$$\min_{\boldsymbol{w}>0} \left| \frac{w_i}{w_j} - a_{ij} \right|_{i \neq j}$$

Former discussions of the optimization.
Choo and Wedley, 2004.
Blanquero, Carrizowa, Conde, 2006.

Pareto optimality (Efficiency)

- In the multi-objective optimization, Pareto optimality can be defined as follows.

  The weight $\boldsymbol{w} > 0$ is Pareto optimal if there is no weights $\tilde{\boldsymbol{w}} > 0$ which satisfies

$$\left|\frac{\tilde{w}_i}{\tilde{w}_j} - a_{ij}\right| \leq \left|\frac{w_i}{w_j} - a_{ij}\right|$$

  for all $i \neq j$, and at least one inequality holds strictly.

- Blanquero, Carrizosa and Conde, 2006.
  - The geometric mean method is Pareto optimal.
  - The eigenvector method may be not Pareto optimal.

An example: the principal eigenvector is not Pareto optimal. (Bozoki and Fulop, 2016)

$$A = (a_{ij}) = \begin{bmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{bmatrix}$$

The principal eigen vector

$$\boldsymbol{w} = \begin{bmatrix} 0.4045179 \\ 0.4361729 \\ 0.1102954 \\ 0.0490138 \end{bmatrix} \to (w_i/w_j) = \begin{bmatrix} 1 & 0.9274255 & 3.6675849 & 8.2531384 \\ 1.0782538 & 1 & 3.9545872 & 8.8989776 \\ 0.2726590 & 0.2528709 & 1 & 2.2502924 \\ 0.1211660 & 0.1123725 & 0.4443867 & 1 \end{bmatrix}$$

$$\tilde{\boldsymbol{w}} = \begin{bmatrix} 0.4361729 \\ 0.4361729 \\ 0.1102954 \\ 0.0490138 \end{bmatrix} \to (\widetilde{w_i}/\widetilde{w_j}) = \begin{bmatrix} 1 & 1 & 3.9545872 & 8.8989776 \\ 1 & 1 & 3.9545872 & 8.8989776 \\ 0.2528709 & 0.2528709 & 1 & 2.2502924 \\ 0.1123725 & 0.1123725 & 0.4443867 & 1 \end{bmatrix}$$

$(\widetilde{w_i}/\widetilde{w_j})$ is closer to (a_{ij}) than (w_i/w_j).

Objective of this study

- To represent Pareto optimality of weights on directed graphs.
  - Pareto optimality of weights is equivalent to strong connectivity of its associated graph (Balanquero, Carrizosa, and Conde, 2006).
  - We give another proof based on elementary graph theory.
- To provide an algorithm inferring Pareto optimal weights.
- To examine how often principal eigenvectors of pairwise comparison matrices are not Pareto optimal.
- To consider the structure of Pareto optimal weights.

A directed graph associated weights

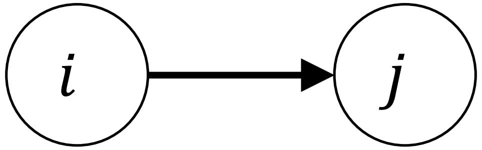
$G(\boldsymbol{w}) \equiv (N, E(\boldsymbol{w}))$, where

$N = \{1, \ldots, n\}$ is the set of nodes corresponding to alternatives.

$E(\boldsymbol{w})$ is the set of directed arcs, where

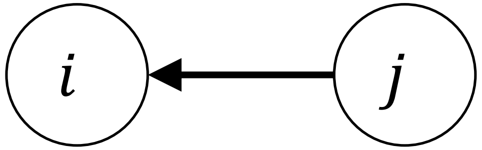
$$E(\boldsymbol{w}) = \left\{(i, j) \mid \frac{w_i}{w_j} - a_{ij} \geq 0\right\}$$

By the definition,
either



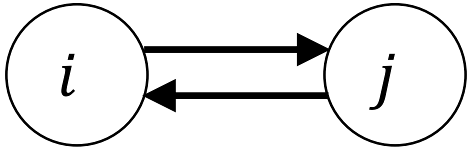
$$\frac{w_i}{w_j} - a_{ij} > 0$$

or



$$\frac{w_i}{w_j} - a_{ij} < 0$$

or



$$\frac{w_i}{w_j} - a_{ij} = 0$$

Pareto optimality and strong connectivity

The weights $\boldsymbol{w}$ is Pareto optimal

$\Updownarrow$

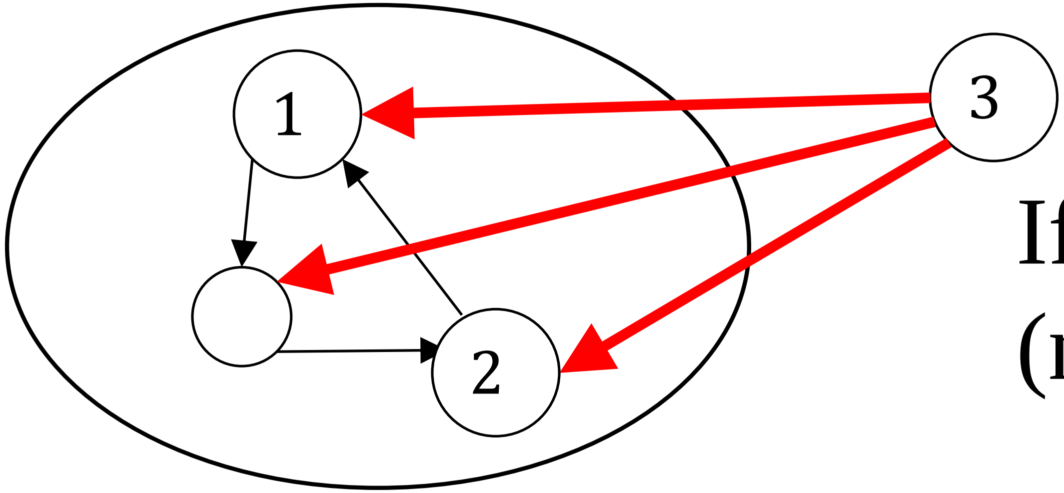
The graph $G = (N, E(\boldsymbol{w}))$ is strongly connected

Strongly connected means that there is a path from i to j for all node pair (i, j).

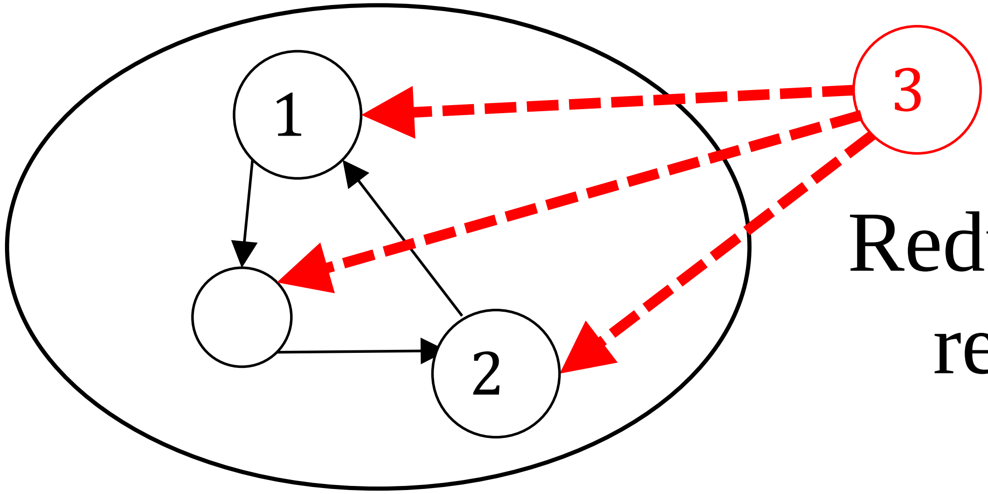
The theorem had shown by Blanquero and Carrizosa, 2006.

We give another proof based on elementary graph theory.

A proof: Pareto optimal ⟹ Strongly connected



If there is no path to node 3,
(not strongly connected)



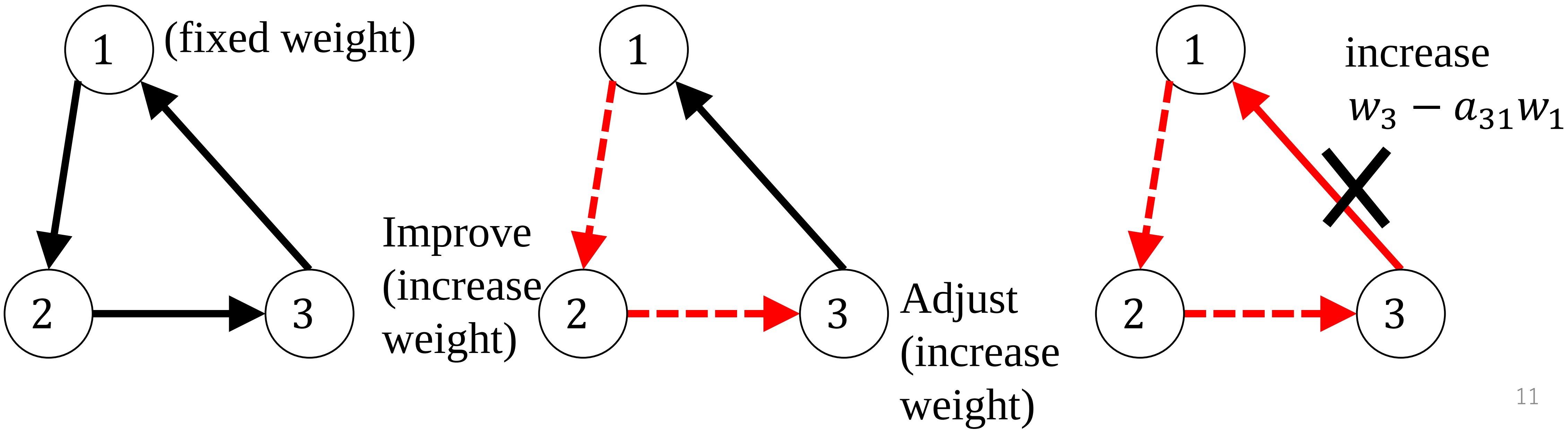
Reducing weight w_3 can
reduces $w_3 - a_{31}w_1$ and $w_3 - a_{32}w_2$

A proof: Strongly connected $\Longrightarrow$ Pareto optimal

If the graph is strongly connected, there is a directed cycle

$$1 \to 2 \to \cdots \to (k-1) \to k \to 1.$$

On the cycle, $\frac{w_i}{w_{i+1}} - a_{i,i+1} > 0$ holds.



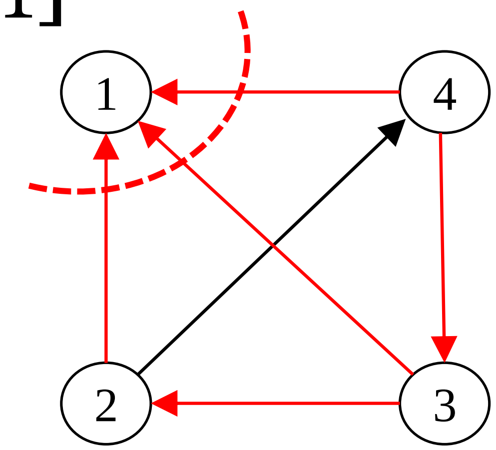
An example

$$A = (a_{ij}) = \begin{bmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{bmatrix}$$

The principal eigen vector

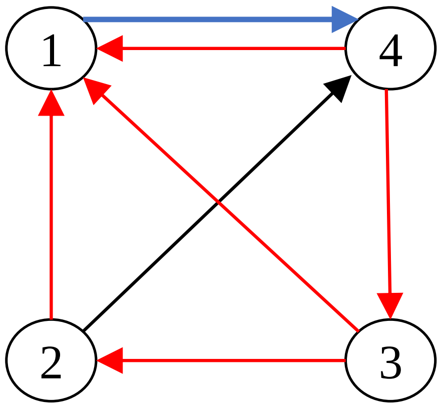
$$\boldsymbol{w} = \begin{bmatrix} 0.4045179 \\ 0.4361729 \\ 0.1102954 \\ 0.0490138 \end{bmatrix} \rightarrow$$

$$(w_i/w_j - a_{ij}) = \begin{bmatrix} 0 & -0.0725745 & -0.3324151 & -0.7468616 \\ & 0 & -3.0454128 & 3.8989776 \\ & & 0 & -1.7497076 \\ & & & 0 \end{bmatrix}$$

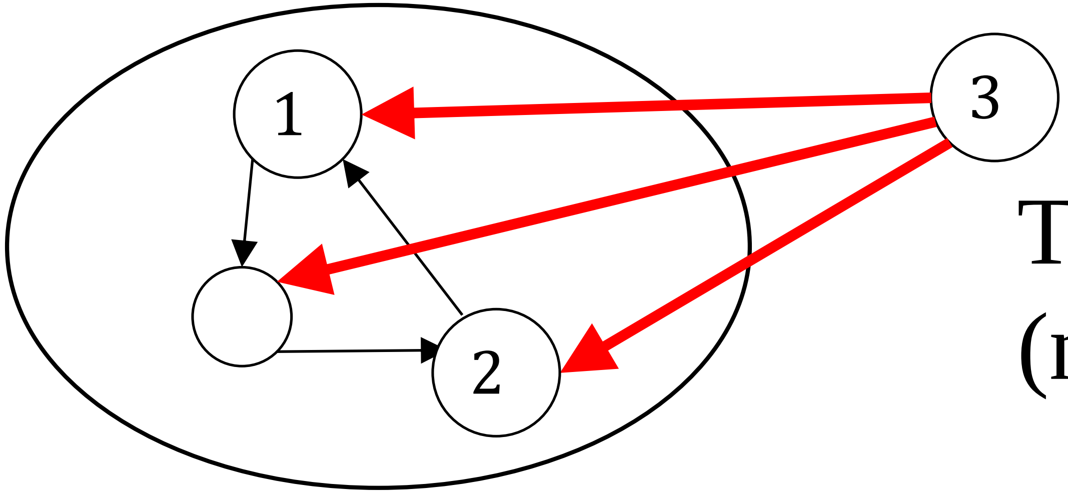


$$\widetilde{\boldsymbol{w}} = \begin{bmatrix} 0.4361729 \\ 0.4361729 \\ 0.1102954 \\ 0.0490138 \end{bmatrix} \rightarrow$$

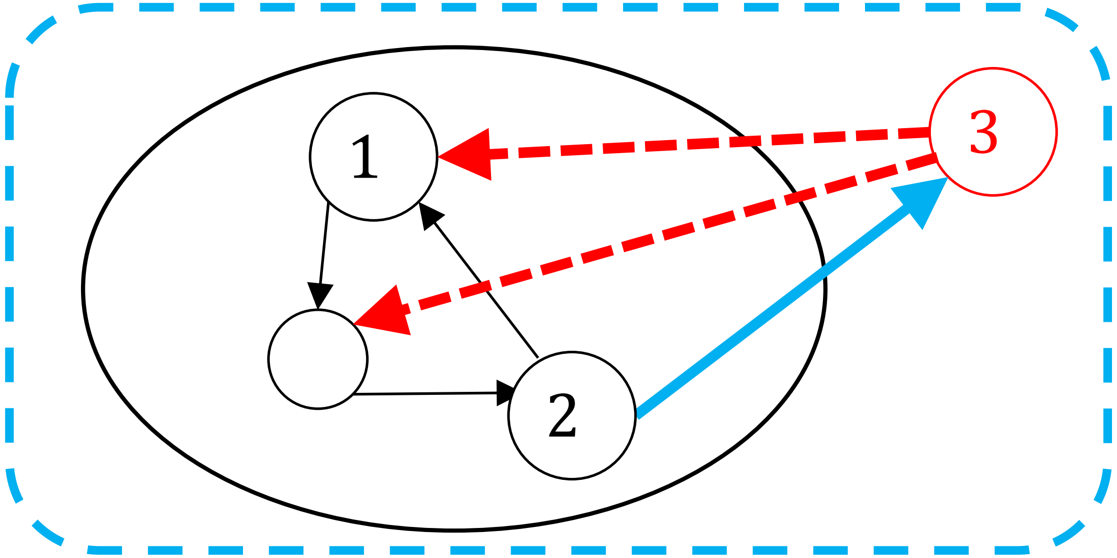
$$(\widetilde{w_i}/\widetilde{w_j} - a_{ij}) = \begin{bmatrix} 0 & 0 & -0.0454128 & -0.1010224 \\ & 0 & -3.0454128 & 3.8989776 \\ & & 0 & -1.7497076 \\ & & & 0 \end{bmatrix}$$



An algorithm to infer weights 1



There is no path to node 3.
(not strongly connected)



Reducing weight w_3 until an arc changes its direction.

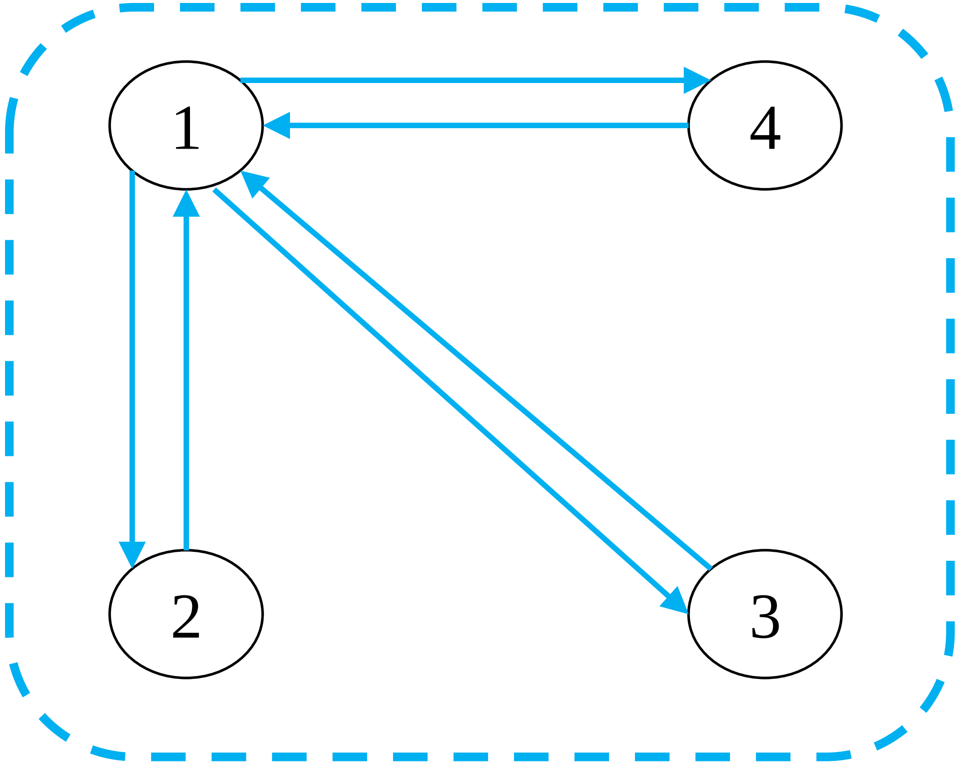
Strongly connected

An algorithm to infer weights 2

Each column of pairwise comparison matrix A is Pareto optimal weights.

$$A = \begin{bmatrix} \mathbf{1} & a_{12} & a_{13} & a_{14} \\ \boldsymbol{a_{21}} & 1 & a_{23} & a_{24} \\ \boldsymbol{a_{31}} & a_{32} & 1 & a_{34} \\ \boldsymbol{a_{41}} & a_{42} & a_{43} & 1 \end{bmatrix}$$

$$\left(\frac{\boldsymbol{a_{1i}}}{\boldsymbol{a_{1j}}} - a_{ij}\right) = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & 0 & \frac{a_{12}}{a_{13}} - a_{23} & \frac{a_{12}}{a_{14}} - a_{24} \\ & & 0 & \frac{a_{13}}{a_{14}} - a_{34} \\ & & & 0 \end{bmatrix}$$



Strongly connected

The geometric mean is always Pareto optimal

It can be proven by making arc (i, j) when

$$\ln w_i - \ln w_j - \ln a_{ij} \geq 0.$$

Geometric mean $\boldsymbol{w}$ is the solution of

$$\sum_{i,j} \left| \ln w_i - \ln w_j - \ln a_{ij} \right|^2 = 0$$

$$\left(\sum_{i \neq k} \ln w_i - \ln w_k - \ln a_{ik} \right) = \left(\sum_{j \neq k} \ln w_k - \ln w_j - \ln a_{kj} \right), \qquad \text{for all } k$$

k is not isolated.

Progresses since submitting abstract

To examine how often the principal eigenvector is not Pareto optimal.

To visualize regions of Pareto optimal weights.

To show nonconvexity of Pareto optimal weights.

How often is the principal eigenvector not Pareto optimal?

Experiment

- For all 4×4 possible 531441 pairwise comparison matrices whose elements are $\left\{\frac{1}{9}, \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, 1, 3, 5, 7, 9\right\}$, we checked Pareto optimality of the principal eigenvector.

Result

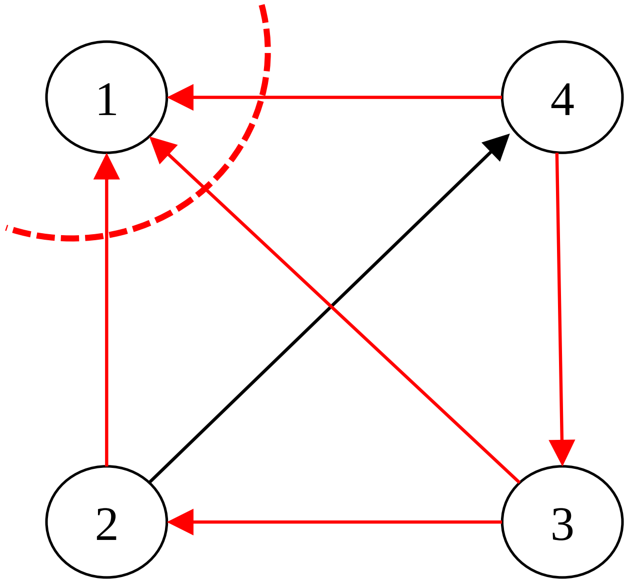
- 75216 (14.15 %) of 531441 eigenvectors are not Pareto optimal.
- 432 (2.31%) of 18681 eigenvectors of matrices whose C.I. < 0.1 are not Pareto optimal.

A typical example

The principal eigenvector

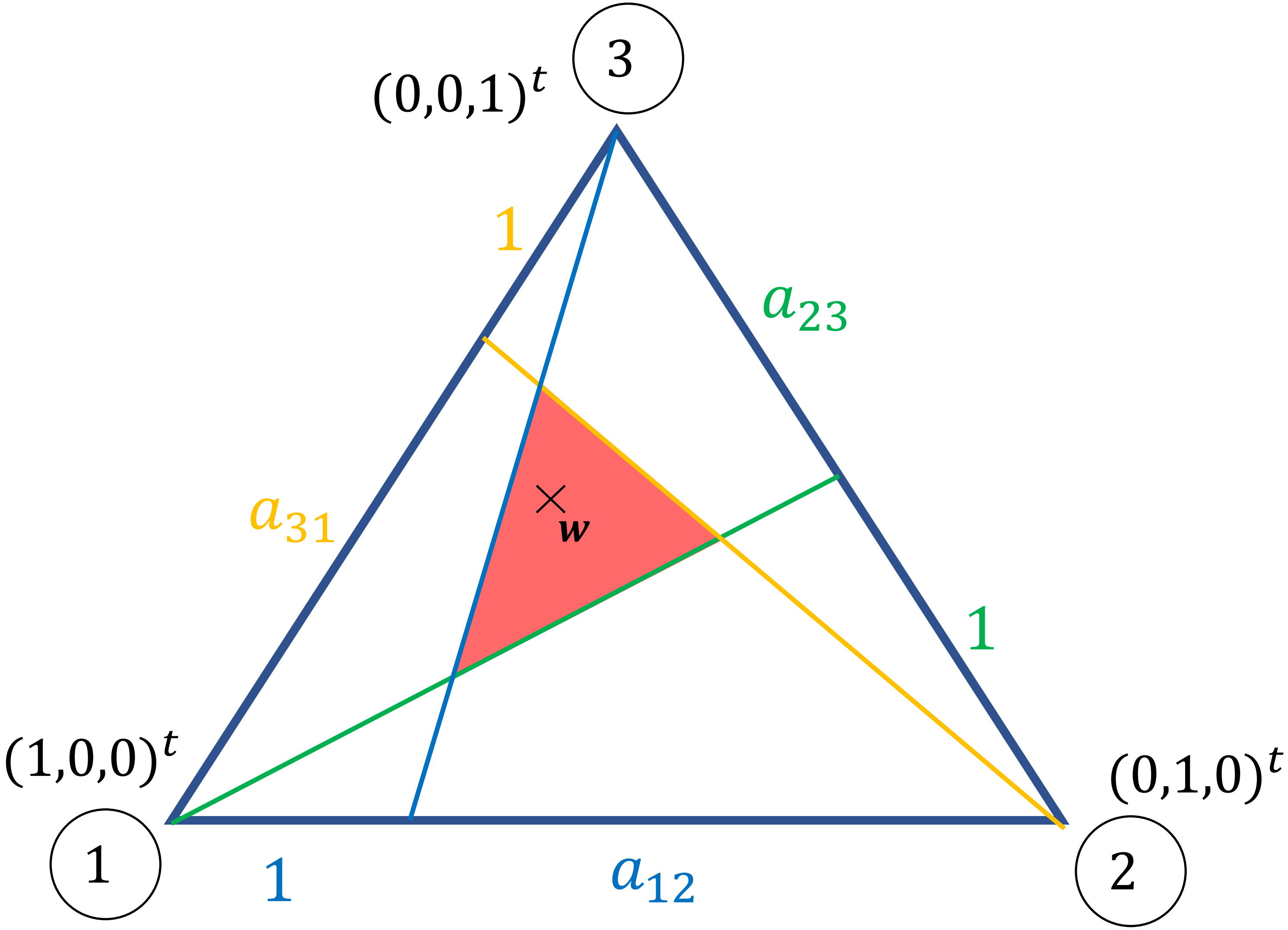
$$A = \begin{bmatrix} 1 & 1 & 3 & 9 \\ 1 & 1 & 5 & 5 \\ 1/3 & 1/5 & 1 & 5 \\ 1/9 & 1/5 & 1/5 & 1 \end{bmatrix}, \ \text{C.I.} = 0.075, \quad \boldsymbol{w} = \begin{pmatrix} 0.3947 \\ 0.4159 \\ 0.1412 \\ 0.0481 \end{pmatrix}$$

$$(w_i/w_j - a_{ij}) = \begin{bmatrix} 0 & -0.0510 & -0.2047 & -0.7942 \\ & 0 & -2.0545 & 3.6466 \\ & & 0 & -2.0644 \\ & & & 0 \end{bmatrix}$$



Visualize the regions of the set of Pareto optimal weight

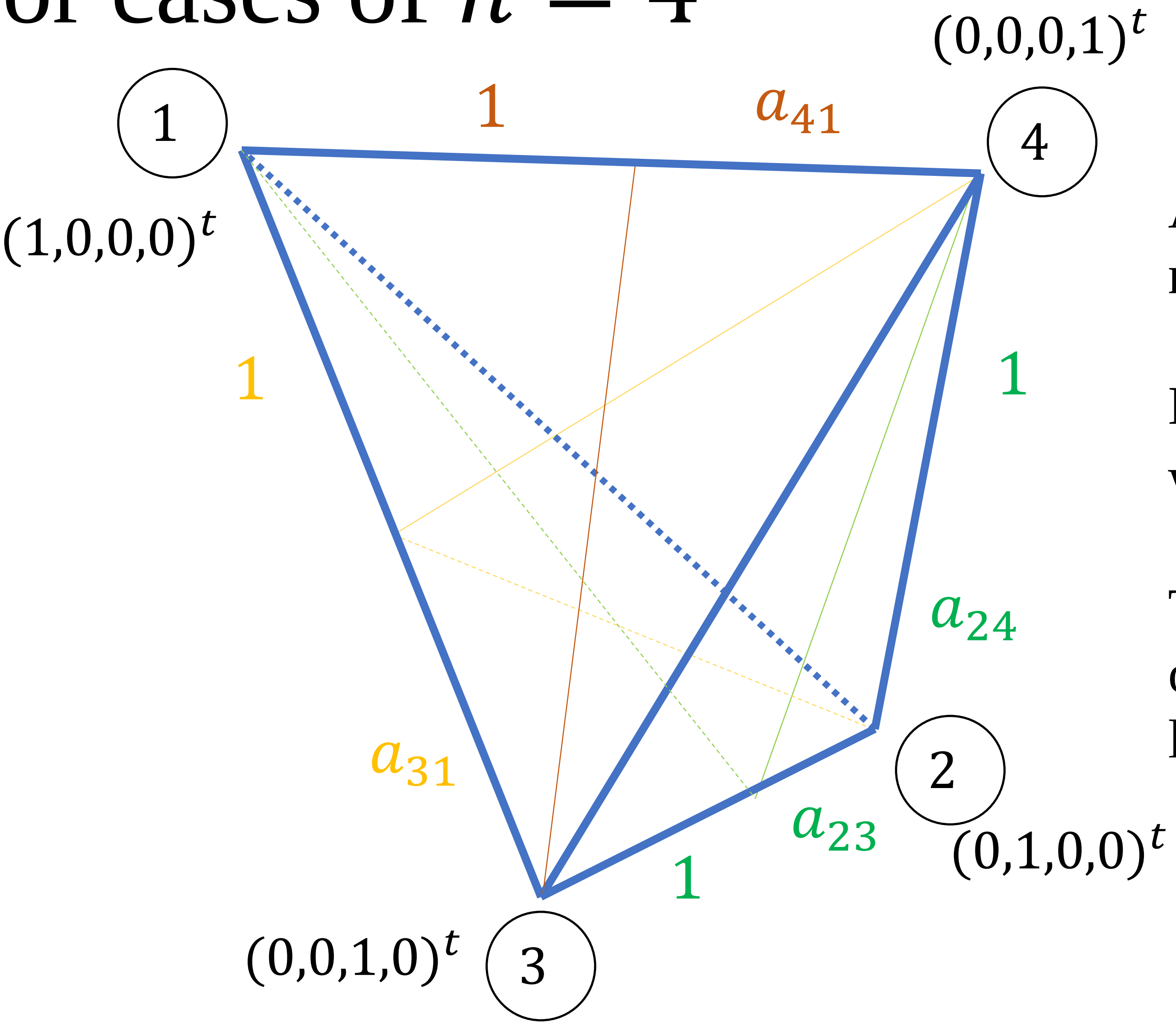
A ternary diagram for cases of $n = 3$. (Mizuno and Taji, 2016)



In cases of $n = 3$,

- If $\boldsymbol{w}$ is in the inner triangle, then $\boldsymbol{w}$ is Pareto optimal.
- The principal eigenvector is Pareto optimal.
- All weights in the inner triangle are Pareto optimal.

For cases of $n = 4$



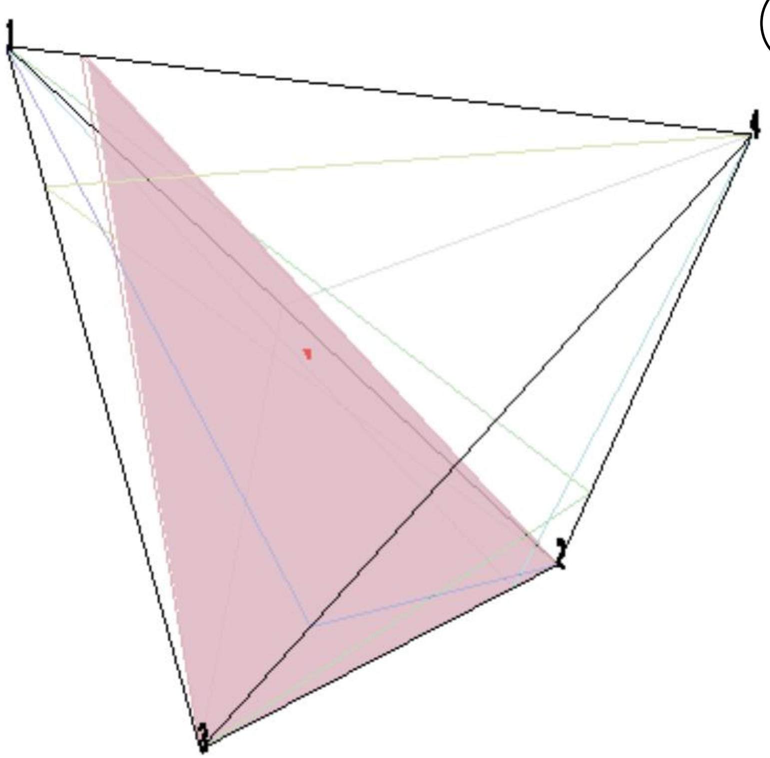
All possible weights are represented in tetrahedron.

Each value a_{ij} is a hyper plane which slices the tetrahedron.

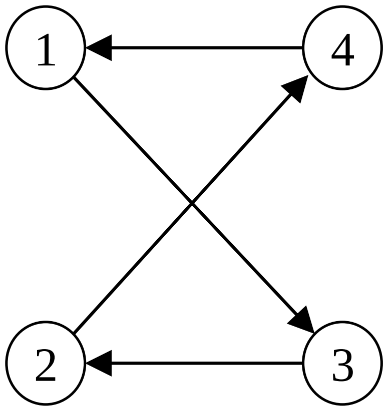
The region of the set of Pareto optimal weights is not any longer convex.

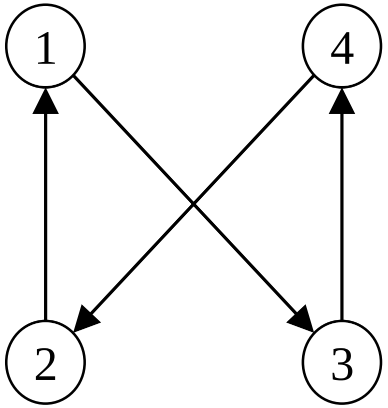
An example

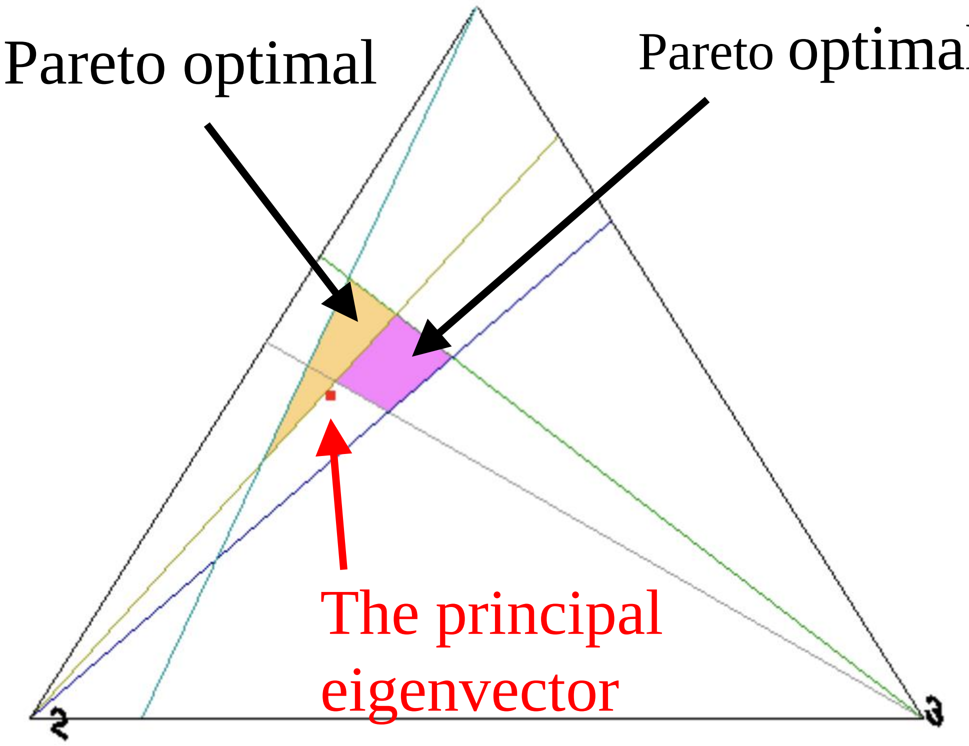
$$\begin{bmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{bmatrix}$$



The plane represents $w_1 : w_4 = 0.404 : 0.049.$



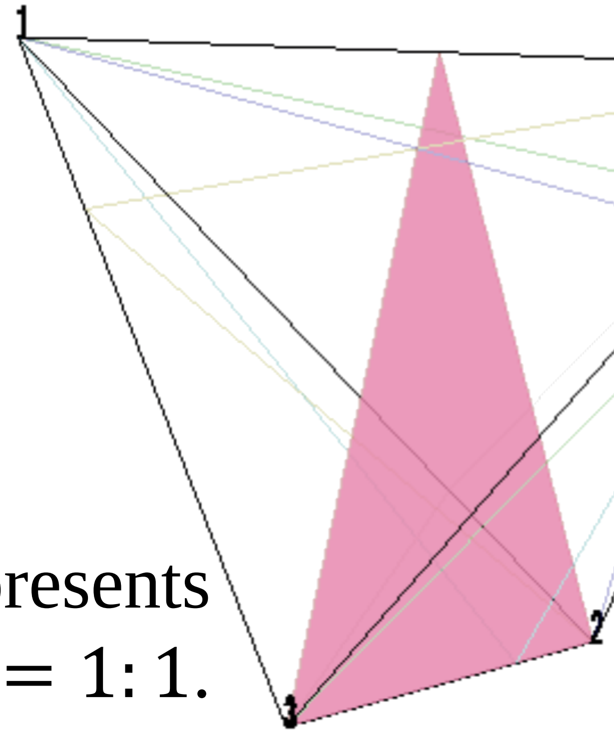


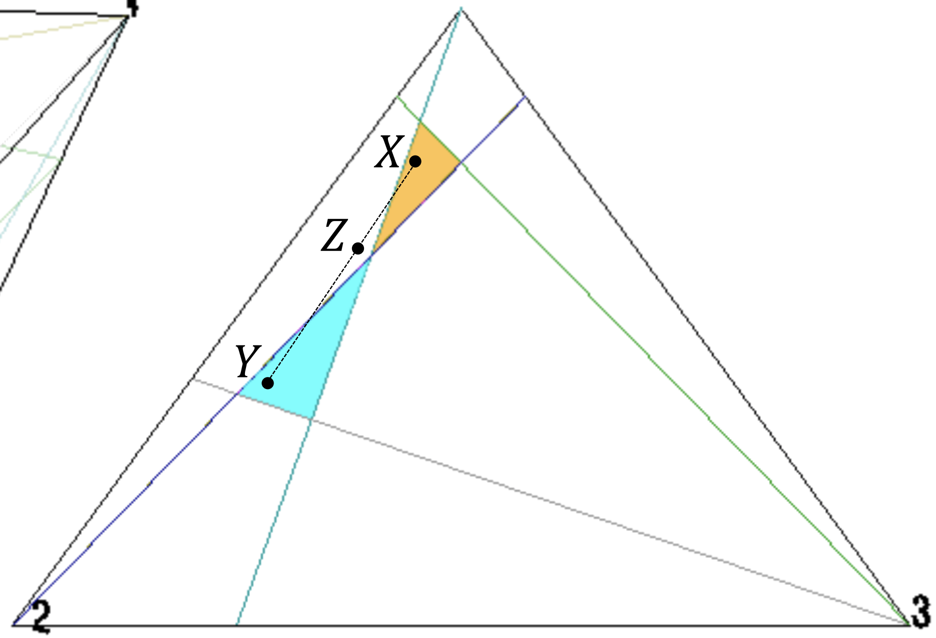


The optimization is nonconvex.

- In case of $N = 3$, Pareto optimal weights construct a convex area.
- In case of $N \geq 4$, it is not convex.

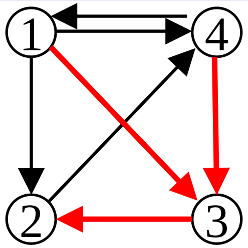
$$\begin{bmatrix} 1 & 1/3 & 3 & 1 \\ 3 & 1 & 3 & 1/3 \\ 1/3 & 1/3 & 1 & 1/3 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

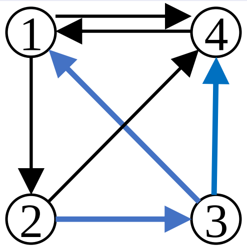


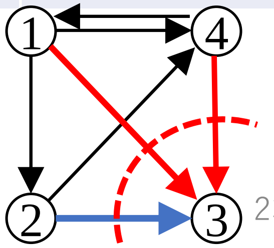


The plane represents $w_1 : w_4 = 1 : 1.$

c	X	+	$(1-c)$	Y	=	Z
$\frac{27}{49} \times$	$\frac{1}{27}\begin{bmatrix}10\\5\\2\\10\end{bmatrix}$	+	$\left(1-\frac{27}{49}\right)\times$	$\frac{1}{22}\begin{bmatrix}5\\10\\2\\5\end{bmatrix}$	=	$\frac{1}{49}\begin{bmatrix}15\\15\\4\\15\end{bmatrix}$
	Efficient			Efficient		Inefficient







Conclusion and Future works

- We provided a proof of the theorem based on elementary graph theory: Pareto optimal weights has directed graphs strongly connected.
- We provide algorithms to infer Pareto optimal weights based on the proof.
- We investigated Pareto optimality of the eigenvector method.
- We visualized the set of Pareto optimal weights.
- The set of Pareto optimal weights are not convex.
  - It consists of convex regions corresponding to strongly connected graphs.
  - Which convex area the geometric mean is in?
- How choose weights from the large regions.