ON PROPERTIES OF PARETO OPTIMAL WEIGHTS FROM PAIRWISE COMPARISON MATRICES BASED ON GRAPH THEORY

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ABSTRACT

Pairwise comparison is frequently used in multi-criteria decision making problems. In the analytic hierarchy process (AHP), the local weights inferred from pairwise comparison matrices by mainly the principal eigenvector method or the geometric mean method. Recently, Blanquero et al. have shown that the weights derived from geometric mean method are Pareto optimal (or efficient) while the eigenvector method is not from the multi-objective optimization view point. They have also shown that the equivalency between Pareto optimal weights and the strong connectivity of the associated directed graphs. In this paper, we first give another proof of the equivalence theorem based on elementary graph theory. Based on the proof, we propose a new method inferring Pareto optimal weights, which is useful to modify the eigenvector method. We also discuss some properties of Pareto optimal weights based on graph theories.

Keywords: AHP, pairwise comparison, Pareto optimality, graph theory.

1. Introduction

Pairwise comparison has been commonly used in decision making problems and various methods have been proposed for inferring the weights of alternatives from the results of pairwise comparisons. In the analytic hierarchy process (AHP) proposed by Saaty in late 1970s, after constructing the hierarchy structure of the problem, pairwise comparisons are performed to evaluate local weights.

In AHP, local weights are considered to be ratio scale, and hence, the pairwise comparison value a_{ij} between the alternatives i and j evaluate the ratio of the weights between w_i and w_j approximately, that is, $a_{ij} \approx w_i/w_j$. The results of pairwise comparisons are summarized in the so-called *pairwise comparison matrix* A, as follows

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & 1 & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 1 \end{pmatrix},$$

where we usually assume that the *reciprocal property*, such that, $a_{ji} = 1/a_{ij}$ holds for $i \neq j$. From pairwise comparison matrices A, there have been various weight inferring methods are proposed. Among these, the principal eigenvector method and the (row) geometric mean method are most famous and widely used, where the principal eigenvector method adopts the local weight vector $w = (w_1, \cdots, w_n)^T$ as the (normalized) principal eigenvector of the pairwise comparison matrix A and the geometric mean method adopts as $w_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}$.

From the optimization point of view, inferring local weights from a pairwise comparison matrix is considered to solve multi-objective optimization problem,

$$\min_{w>0} \left| \frac{w_i}{w_j} - a_{ij} \right|_{i,j=1,\cdots,n,i\neq j}.$$

In this formulation, the notion of Pareto optimality (or efficiency) is practically and theoretically important, where the local weight $w > 0$ is *Pareto optimal* if there is no positive vector $\widetilde{w} > 0$, such that, the inequalities $\left|\widetilde{w}_i/\widetilde{w}_j - a_{ij}\right| \leq \left|w_i/w_j - a_{ij}\right|$ hold for $i \neq j$ and at least one inequality holds strictly.

Recently, Blanquero ed al. have proposed the LP based method for testing the Pareto optimality of the given local weight w. They also have given the equivalence theorem between the Pareto optimality and the strong connectivity of the associated directed graph, but they have not given methods asserting Pareto optimality of local weights.

In this paper, we first give another proof of the equivalent theorem proven by Blanquero ed al. The proof is strongly depend on the graph theory, and, based on the proof; we propose some methods for inferring Pareto optimal weights from pairwise comparison matrices, which are also shown to be practically useful for making the principal eigenvector Pareto optimal. Furthermore, we give some properties based on the above mentioned elementary proof, and discuss the relationship between the eigenvector method and Pareto optimality by some numerical experiences.

2. Preliminaries

In this section, we summarize some notations and some properties from graph theories used in this paper. A graph $G = (N, E)$ consists of the set of nodes $N = \{1.2. \cdots, n\}$ and the set of arcs $E = \{(i,j) \in N \times N\}$. A graph is called *directed graph* if the arcs (i,j) and $(j,i) \in E$ are distinguished and called *undirected graph* if otherwise. A *directed path* connecting from a node i to a node j consists of a sequence of directed arcs $(i,k_1), (k_1,k_2), \cdots, (k_l,j) \in E$, which is called undirected path (or simply called a path) if these directions of arcs are ignored. A (directed) *cycle* is a (directed) path if two nodes i and j are identical. A (directed) path is called *simple* if it does not contain a cycle. A directed graph is called *strongly connected* if there are directed paths between all node pairs $(i,j) \in N \times N$, and called *weakly connected* if there exist undirected paths between all node pairs. A weakly connected directed graph is often called *complete*.

A graph $G' = (N', E')$ is called a *subgraph* of a graph G if $N' \subseteq N$ and the set of arcs E' consists of all arcs $N' \times N'$ in E. If a subgraph G' is strongly connected, we call it a *strongly connected component*. It is known that strongly connected components of a complete directed graph make a lattice with the maximum element.

3. Main Results

In this section, we first show the proof of the equivalence between a Pareto optimal local weight $w > 0$ and the strong connectivity of the associated directed graph. This equivalence has already shown by Blanquero et al., but our proof is based on elementary graph theory and then proposes a method deriving Pareto optimal weights from a pairwise comparison matrices based on the proof.

For a given $n \times n$ pairwise comparison matrix A and a positive vector $w > 0$, we construct a directed graph $G(w) = \big(N, E(w)\big)$, such that, the node set is $N = \{1,2,\cdots,n\}$ and the arc set $E(w) \subseteq N \times N$ includes an arc (i,j) if and only if $w_i/w_j - a_{ij} \geq 0$. From the construction of the graph $G(w)$, there exists either a directed arc (i,j), (j,i) or both, and hence, the graph $G(w)$ is weakly connected (or complete) for all positive $w > 0$.

Theorem 1. For a given pairwise comparison matrix $A = \big(a_{ij}\big)$ and a given positive vector $w = (w_1, \cdots, w_n)^T > 0$, the weights w is Pareto optimal for an associated pairwise comparison matrix A if and only if the directed graph $G(w)$ is strongly connected.

Proof. (only if part) Suppose that the graph $G(w)$ is not strongly connected. Then there exists nodes $i,j \in N$, such that there is no directed path from i to j in $G(w)$, and more generally, the node set N is partitioned into two sets N_i and $\overline{N}_i$, such that, $N_i \cup \overline{N}_i = N$, $N_i \cap \overline{N}_i = \emptyset$ and N_i is the node set having a direct path from the node i (including i itself) and $\overline{N}_i$ is not. From the construction rule of $G(w)$, inequalities $w_i/w_j - a_{ij} < 0$ hold for $i \in N_i$ and $j \in \overline{N}_i$. Setting the positive parameter t as

$$t = \max\left\{\frac{w_i}{a_{ij}w_j}\,\middle|\, i \in N_i \text{ and } j \in \overline{N}_i\right\},$$

and making a new positive vector $\overline{w} > 0$, such that, $\overline{w}_i = w_i/t$ for $i \in N_i$ and $\overline{w}_j = w_j$ for $j \in \overline{N}_i$. Then, it is easy to see that $|\overline{w}_k/\overline{w}_l - a_{kl}| = |w_k/w_l - a_{kl}|$ holds for $k,l \in N_i$ or $k,l \in \overline{N}_i$ and $w_i/w_j - a_{ij} < \overline{w}_i/\overline{w}_j - a_{ij} < 0$ holds for $i \in N_i$ and $j \in \overline{N}_i$, because $0 < t < 1$. This shows that the weight vector w is not Pareto optimal.

(if part) Suppose that the graph $G(w)$ is strongly connected but the weight vector w is not Pareto optimal. Then without the loss of generality, we can assume as follows. There is a weight vector $\overline{w} > 0$ where $\left|\overline{w}_i/\overline{w}_j - a_{ij}\right| \leq \left|w_i/w_j - a_{ij}\right|$ holds for $i,j \in N$ and $|\overline{w}_1/\overline{w}_2 - a_{12}| < |w_1/w_2 - a_{12}|$ with $\overline{w}_1 = w_1$ (This holds because the weight vector is unique up to positive multiple), and there is a simple directed cycle such as $(1,2), (2,3), \cdots, (k-1,k), (k,1)$.

From the above assumptions, the inequality $\overline{w}_2 > w_2$ must hold because $w_1/w_2 - a_{12} > 0$ by the existence of a directed arc $(1,2)$. To keep the Pareto optimality condition, this leads to the inequality $\overline{w}_3 > w_3$. Same arguments show that $\overline{w}_i > w_i$ must hold for $i = 2,3,\cdots,k$, in particular $\overline{w}_k > w_k$, but this is impossible because we have $0 < w_k/w_1 - a_{k1} < \overline{w}_k/\overline{w}_1 - a_{k1}$. Q.E.D.

Let consider the case that the pairwise comparison value a_{ij} represent an interval scale, that is, a_{ij} approximates the differences between w_i and w_j, $a_{ij} \approx w_i - w_j$. In this case, if the construction of the graph $G(w)$ is modified such that an arc (i,j) is in $E(w)$ if and only if $w_i - w_j - a_{ij} \geq 0$, then the similar argument of the above proof shows the equivalence between the strong connectivity of $G(w)$ and the Pareto optimality of the weight vector w (Of course the definition of Pareto optimality should be changed according to the interval scale). This also shows the Pareto optimality of the geometric mean method.

Corollary 2. The positive weight vector $w > 0$ derived by the (row) geometric mean method from a pairwise comparison matrix $A = (a_{ij})$ is Pareto optimal.

From the proof of the theorem 1, in particular from the only if part, we can construct several methods for deriving a local weight vector $w > 0$ from a pairwise comparison matrix $A = (a_{ij})$. The following algorithm is only one candidate of these.

Algorithm

Step 0: Set the initial positive vector $w > 0$, and construct the associated directed graph $G(w)$.

Step 1: If $G(w)$ is strongly connected, then STOP. The current weight w is Pareto optimal. Otherwise, select two strongly connected components $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$ (Without the loss of the generality, we assume that the inequalities $w_i/w_j - a_{ij} < 0$ hold for $i \in N_1$ and $j \in N_2$).

Step 2: Select a positive parameter $t > 0$ from the interval

$$\min\left\{\frac{w_i}{a_{ij}w_j} \,\middle|\, i \in N_1 \text{ and } j \in N_2\right\} \leq t \leq \max\left\{\frac{w_i}{a_{ij}w_j} \,\middle|\, i \in N_1 \text{ and } j \in N_2\right\},$$

and set $w_i \leftarrow w_i/t$ for $i \in N_1$. Return to Step 1.

There are some freedoms to implement the above algorithm. In particular, how to choose a parameter $t > 0$ in Step 2 influences the performance and the results of the algorithm. We call setting $t = \max\{w_i/a_{ij}w_j\}$ as minimal modification, $t = \min\{w_i/a_{ij}w_j\}$ as maximal modification, and $t = (\max\{w_i/a_{ij}w_j\} + \min\{w_i/a_{ij}w_j\})/2$ as average modification.

The results of some numerical experiments will be shown in the oral presentation.

4. Concluding Remarks

In this paper, we have shown an equivalent theorem of a Pareto optimal weight and the strong connectivity of the associated directed graph. The theorem itself has been already shown by Blanquero et al. but our proof is based on elementary graph theory and it gives an algorithm ensuring Pareto optimal local weight vectors.

From our graph theoretical analysis, we can derive some properties of Pareto optimal vectors for a given pairwise comparison matrix, and also we have some open problems mainly in the view of convex analysis. We list some of these as following.

- From the if part of the proof of the theorem, it is easily to show the weakly Pareto optimal theorem, that is, the weight vector $w > 0$ is weakly Pareto optimal if and only if the graph $G(w)$ has no directed cycle.
- We can also show the boundedness of the normalized weight vector (e.g. $w_1 = 1$ is fixed) by the same arguments in the if part of the theorem.
- For a given Pareto optimal weight vector $w^* > 0$, the region of the weight vectors w having the same directed graph $G(w^*)$ (this means that all direction of the arcs are

same) forms a polyhedral convex cone (without the origin) in the nonnegative orthant. But whether the all Pareto optimal form a convex cone is not known for the case of $n \geq 4$.

- By using the ternary diagram proposed by Mizuno and Taji, Pareto optimal vectors form a triangle (and hence, convex region) in a ternary diagram for the case of $n = 3$.But this also shows that the region of Pareto optimal vectors is relatively large. So which should be selected from the Pareto optimal region is one of important practical issue.

5. Key References

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