AHP : The signal and the noise – A numerical experiment on the possibility of getting the solution with much less pairwise comparisons

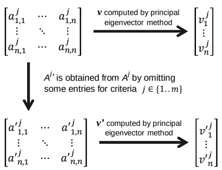
Robin Rivest, M.Sc. Candidate at HEC Montreal,
Montreal, Quebec, Canada

Introduction

The number of comparisons required to fill the pairwise comparison (PC) matrix used in the scientific study of preferences and in particular in the AHP can become tedious as the number of alternatives considered becomes larger (grows with $O(N^2)$).

Priority vectors which are obtained from normalizing principal eigenvectors of **PC** matrices can be computed even if some **PC** entries are missing, under some conditions.

This study aims to determine whether or not some **PC** matrices entries can be systematically omitted in the elicitation process of the AHP without significantly distorting the final solution. It is expected that these omissions will be guided by a number of simple heuristics that will have been verified empirically by way of numerical simulations. The simulations compare priority vectors obtained from complete matrices with those obtained by omitting some PC entries (see figure 1).



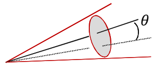
– Figure 1 –
Computation of priority vector

The measure used to evaluate distances between priority vectors is the angle based on the cosine similarity of vectors which is defined as :

$$d(\boldsymbol{v},\boldsymbol{v'}) = \theta = \cos^{-1}\left(\frac{v \circ v'}{\sqrt{v \circ v \cdot v' \circ v'}}\right)$$

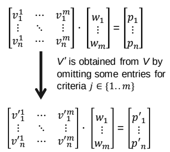
Core concepts and computations

- The space of priority vectors obtained from filling ***PC*** matrices using scales such as Saaty's linear, Ma-Zeng's inverse linear, Hamalainen's balanced, etc., is discrete.
- All random vectors $\boldsymbol{w}=(w_1, \ldots, w_n)$ used to fill a ***PC*** matrix by computing $\boldsymbol{A}$ ($a_{i,j} = round\,[w_i / w_j]$), which result in the same priority vector (e.g. by increments of 0.001) are included within a cone around the resulting priority vector.
  - Therefore, the use of discrete scales entails a <u>limit of precision</u> for priority vectors which can be expressed in angles using the cosine similarity measure (see figure 2).



– Figure 2 –
Illustration of the cone of precision for a priority vector of dimension 3.

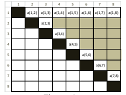
- Solution vector ***p'*** obtained with **PC** matrices omitting a number of entries is an approximation of the solution vector ***p*** obtained with complete matrices (see figure 3):
  - If the approximation error measured as d(***p***,***p'***) is smaller than the <u>limit of precision</u>, ***p'*** can be considered just as accurate.



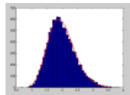
– Figure 3 –
V is the N X M matrix concatenating the M priority vectors (one for each criteria).

Preliminary Results

- Preliminary tests (as of this submission deadline – April 18, 2016) have shown :
  - That priority vectors obtained from matrices of sizes 5 X 5 to 15 X 15 have a limit of precision angle of approximately 5 degrees (95th percentile)
  - That overall solutions with models of 6 to 8 alternatives and 4 to 8 criteria with incomplete matrices using a simple heuristic focusing on <u>2N - 3</u> entries (see figures 4a, 4b) provide result vectors within less of 5 degrees from solutions obtained with complete matrices



– Figure 4a –
Rank order the alternatives, then proceed by comparing the following pairs :
- Those including the top alternative
- Those consisting of contiguous alternatives



– Figure 4b –
Distribution of angle distance (In degrees) from a run a 10 000 trials

- Mixed Integer Nonlinear Programming models have shown that the minimum numbers of entries required to approximate (within 5 degrees) the complete solutions of size N can be much less than 2N - 3 (as small as N + 2 in some cases)

- Further tests are being conducted :
  - With other preference scales (Inverse Linear, Balanced, ..)
  - To determine and evaluate other heuristics that could provide approximation with a lesser number of entries
  - To determine the extent to which approximations are affected by inconsistency
  - To confirm the conditions under which the heuristics can be applied

Conclusions

Preliminary results point towards the affirmation that the number of entries required to approximate overall solutions within limits of precision grow with O(N) (see table 1).

N	N*(N-1) / 2	2*N - 3	% entries saved
5	10	7	30%
6	15	9	40%
7	21	11	48%
8	28	13	54%
9	36	15	58%
10	45	17	62%
11	55	19	65%
12	66	21	68%
13	78	23	71%
14	91	25	73%
15	105	27	74%

– Table 1 –
Number of PC matrix entries that can be omitted

Further testing will help specify the conditions under which the heuristics could provide a significant reduction of effort in the elicitation of pairwise comparisons.

Literature references

Harker, P.T., 1987, "Incomplete Pairwise Comparisons in the Analytic hierarchy Process", Mathematical Modelling, 9(11), 837-848.

Zahir, S., 1999, "Geometry of decision making and the vector space formulation of the analytic hierarchy process", European Journal of Operational Research, 112(2), 373-396

Ishizaka A., Labib A., 2011, "Review of the main developments in the analytic hierarchy process", Expert Systems with Applications, 38(11), 14336-14345.

Bozóki, S., Dezső, L., Poesz, A., Temesi, J., 2013, "Analysis of pairwise comparison matrices: an empirical research", Annals of Operations Research 211(1), pp.511–528.

Acknowledgments

Thanks firstly to my dissertation advisor, Erick Delage, Associate Professor, Department of Decision Sciences, HEC Montreal and Holder of the Canada Research Chair in Decision Making under Uncertainty.

Also thanks to Colin Purrington for providing poster templates and poster design guidelines.

Draft version 2016 / 04 / 18