EFFICIENCY

ABSTRACT

A weight vector is called efficient (Pareto optimal) if no other weight vector is at least as good in approximating the elements of the pairwise comparison matrix, and strictly better in at least one position. The least squares method and the logarithmic least squares method always yield efficient weight vectors, while the principal right eigenvector can be inefficient. The talk summarizes some recent results and open questions on efficiency. Lecture slides can be downloaded at http://www.sztaki.mta.hu/~bozoki/slides

Keywords: efficient weight vector, Pareto optimal, principal right eigenvector

1. Introduction

Let pairwise comparison matrix $\mathbf{A} = (a_{ij})$ of size $n \times n$ be given.

Definition A weight vector **w** is *efficient*, if no other weight vector **v** exists such that $|a_{ij} - v_i/v_j| \leq |a_{ij} - w_i/w_j|$ for all $1 \leq i, j \leq n$ and there exist $1 \leq k, l \leq n$ such that $|a_{kl} - v_k/v_l| < |a_{kl} - w_k/w_l|$.

Almost thirty years after the concept of pairwise comparison matrix had been introduced, Blanquero, Carrizosa and Conde (2006) observed that the principal right eigenvector can be inefficient. They also proved that the geometric mean, i.e., the optimal solution to the logarithmic least squares problem, is efficient. Similarly, the optimal solution(s) to the least squares problem are efficient.

2. Main results

Theorem (Bozóki, 2014): The principal right eigenvector of the pairwise comparison matrix

B =

1	p	p	p	$\cdots$	p	p	p
$1/p$	1	q	1	$\cdots$	1	1	$1/q$
$1/p$	$1/q$	1	q	$\cdots$	1	1	1
$1/p$	1	$1/q$	1	$\cdots$	1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$1/p$	1	1	1		1	q	1
$1/p$	1	1	1	$\cdots$	$1/q$	1	q
$1/p$	q	1	1	$\cdots$	1	$1/q$	1

is inefficient. The *CR* inconsistency of **B** can be arbitrarily small if q is close enough to 1.

Theorem (Ábele-Nagy, Bozóki, 2016): If a pairwise comparison matrix can be made consistent by a modification of a single element and its reciprocal, then the principal right eigenvector is efficient.

Theorem (Ábele-Nagy, Bozóki, Rebák, 2016): If a pairwise comparison matrix can be made consistent by a modification of two elements and their reciprocal, then the principal right eigenvector is efficient.

It is somehow surprising that even the average of weight vectors calculated from spanning trees (Tsyganok 2000, 2010, Siraj, Mikhailov and Keane 2012a, 2012b) can be inefficient.

Theorem (Bozóki, Fülöp, 2016): Any inefficient weight vector can be improved in finite number of steps. The linear program, given in details in the paper cited, yields an efficient result.

3. Open questions

Special cases of pairwise comparison matrices have been analyzed regarding the efficiency of the principal right eigenvector. However, the general case is still open, namely the characterization of the principal right eigenvector's efficiency. The generalization of efficiency to incomplete pairwise comparison matrices is similarly inspiring.

4. Key References

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