Judgment Scales of the Analytic Hierarchy Process

The Balanced Scale

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The Balanced Scale

- No judgment, whether this balanced scale (or others) are better or worse than the fundamental AHP scale
- Highlight a correction/generalization of the balanced scale
- This presentation is a part of an article about AHP scales, submitted for publication

AHP Scales

- Fundamental AHP scale uses integers 1, 2, 3 …9 or their verbal equivalents
- Derived from the psychophysical law of Weber–Fechner
- Several other numerical scales have been proposed
- The balanced scale was proposed by Salo & Hämäläinen in 1997

AHP Scales

- Simple case of two criteria:

$$w_{\text{AHP}} = \frac{r}{r+1} \qquad (1)$$

with r = ratio

- We introduce a scale function c

$$r = c(x)$$

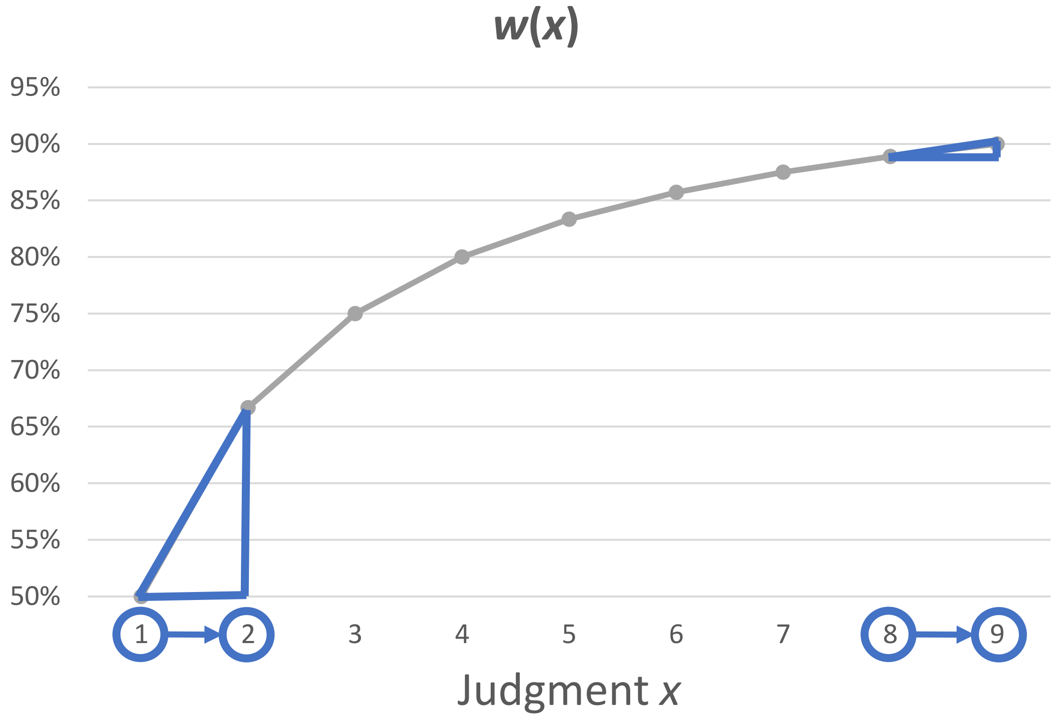
- AHP fundamental scale function

$$c(x) = x$$

- x are the pairwise comparison judgments.
- c resp. $1/c$ are the entry values into the decision matrix and

The Fundamental AHP Scale

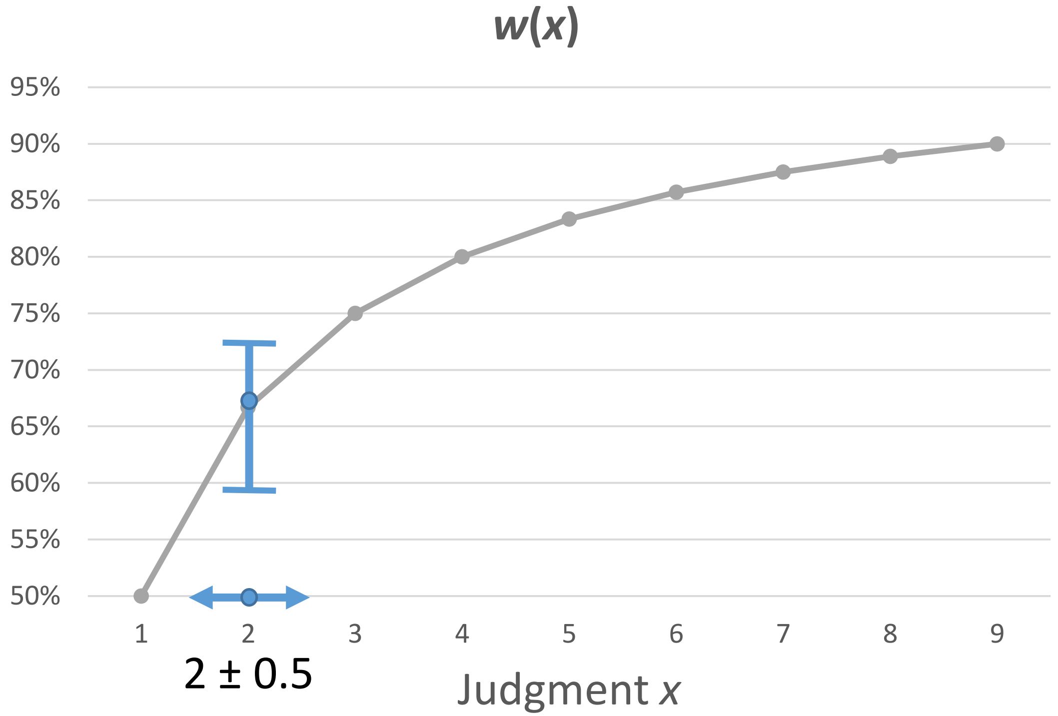
- AHP Weights as function of judgments *x* (1 … 9)



- A change from $x = 1$ to $x = 2$ yields to Δw_{AHP} of 17%
- A change from $x = 8$ to $x = 9$ yields to Δw_{AHP} of 1.1%
- A difference by a factor of 15
- There is a lack of sensitivity, when comparing elements close to each other.

The Fundamental AHP Scale

- AHP Weights as function of judgments *x* (1 … 9)



- Weight uncertainty due to "quantization" of *x* ± 0.5
- A judgment of *x* = 2 results in a local priority of

$$w_{AHP} = (67^{+5}_{-7})\%$$

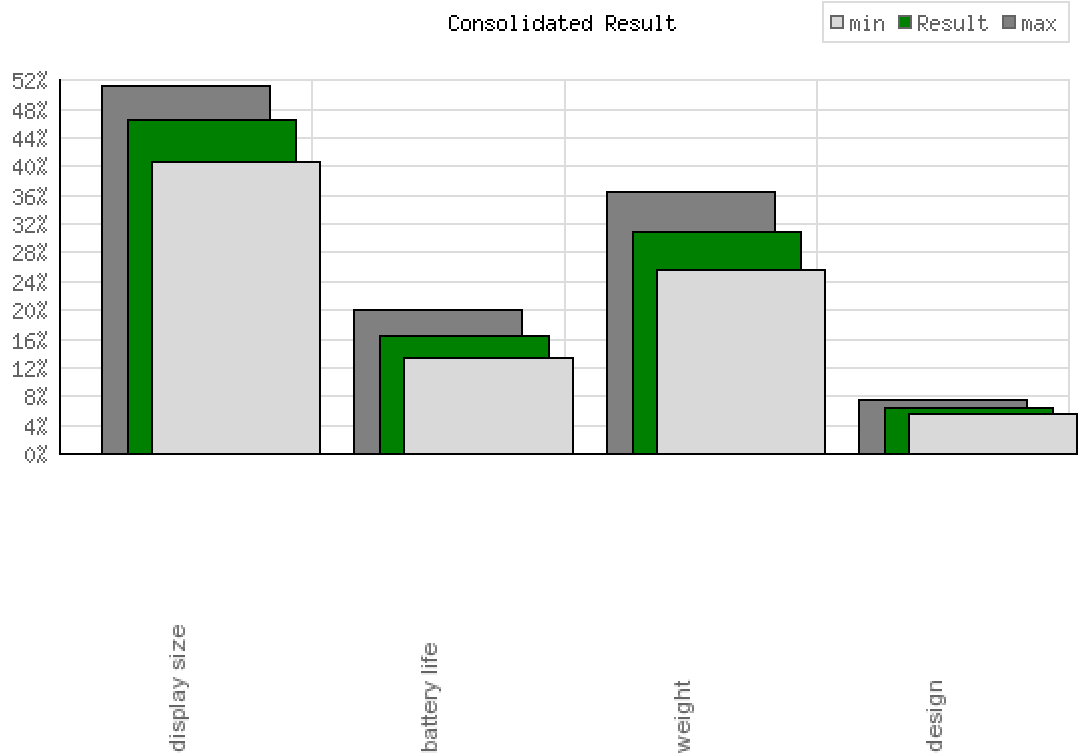
The Fundamental AHP Scale

- Example

Decision Hierarchy		
Level 0	Level 1	Glb Prio.
Buy tablet computer	display size 0.464	46.4%
	battery life 0.163	16.3%
	weight 0.308	30.8%
	design 0.065	6.5%
		1.0

Group result	46.4%	16.3%	30.8%	6.5%	2.8%
(+)	4.8%	3.6%	5.7%	0.9%	n/a
(-)	5.8%	2.9%	5.0%	0.8%	n/a

- Uncertainties



Salo & Hamalainen (1997)

Salo & Hamalainen (1997) introduced the balanced scale using:

$$w_{\mathrm{bal}} = 0.45 + 0.05\,x$$

w_{bal} = 50%, 55%, 60% … 90%
for *x* = 1, 2, 3, … 9

$$c = \frac{w_{\mathrm{bal}}(x)}{1 - w_{\mathrm{bal}}(x)}$$

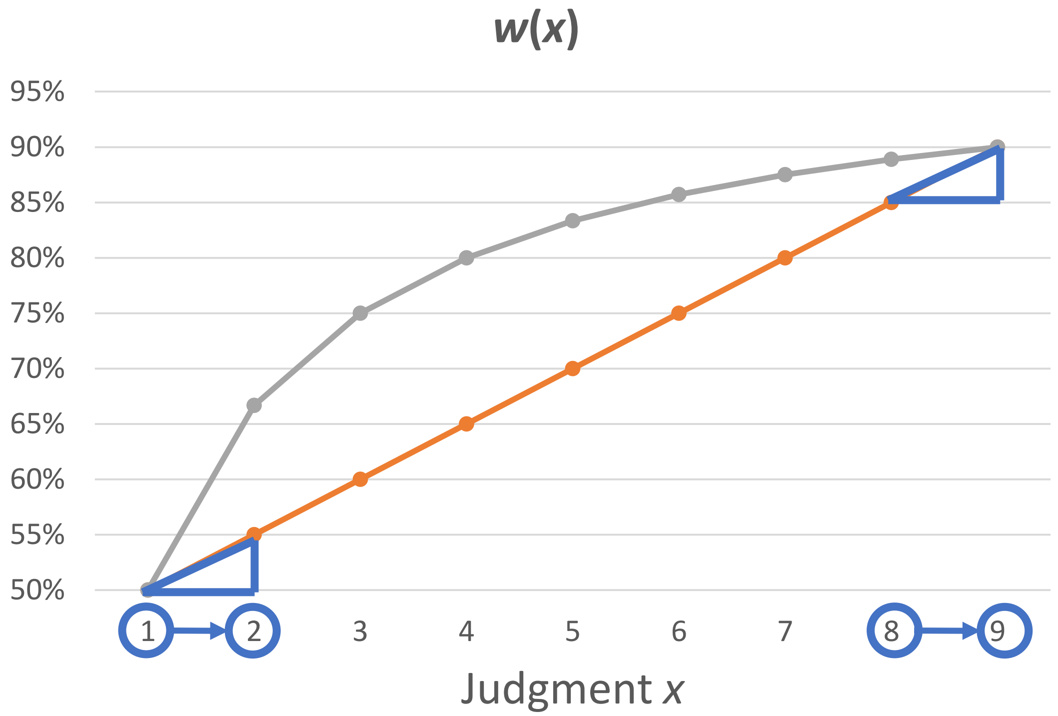
- The Balanced Scale can be written as

$$c = \frac{9 + x}{11 - \mathrm{x}}$$

- *c* resp. 1/*c* are the entry values into the decision matrix and
- *x* the pairwise comparison judgments.

Salo & Hamalainen (1997)

- AHP Weights for the balanced scale (2 criteria)



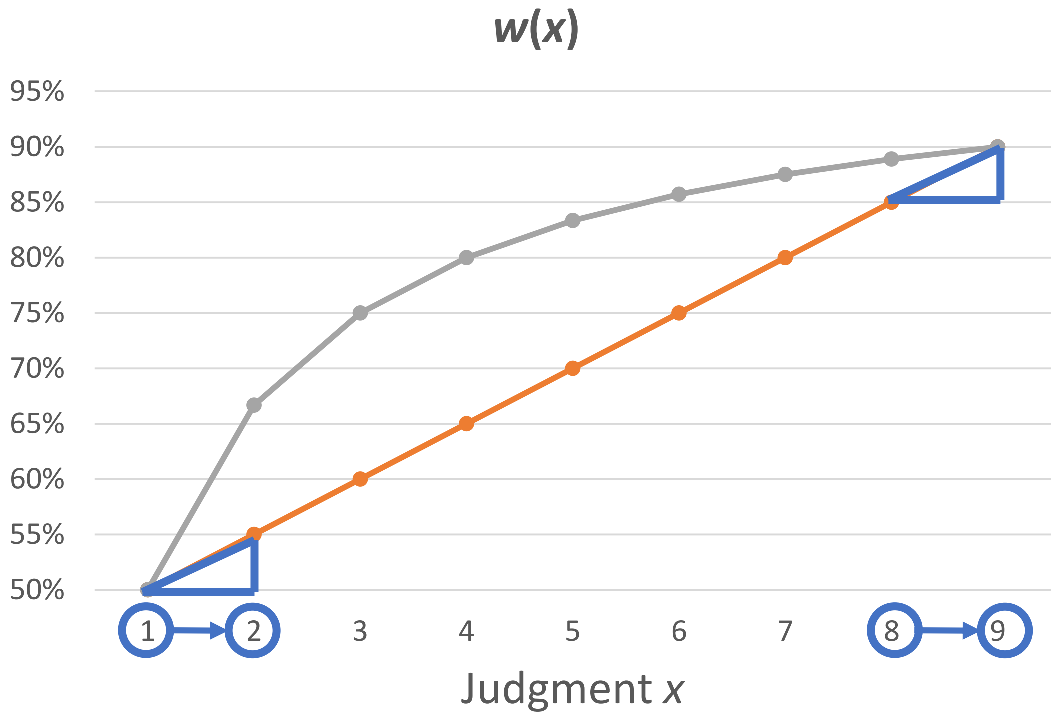
- The Balanced Scale can be written as

$$c = \frac{9 + x}{11 - x}$$

- c resp. $1/c$ are the entry values into the decision matrix and
- x the pairwise comparison judgments.

Salo & Hamalainen (1997)

- AHP Weights for the balanced scale (2 criteria)



- Weight uncertainty due to "quantization" of x ± 0.5 is const over the whole judgment range.

The Generalized Balanced Scale

$$w_{\mathrm{AHP}} = \frac{r}{r+1} \qquad (1)$$

(1) is a *special case* for one pairwise comparison of *two criteria*!

$$w_{\mathrm{AHP}} = \frac{r}{r+n-1} \qquad (2)$$

(2) is the generalized case for *n* criteria

- Normalized geometric mean of the first row

$$DM = \begin{pmatrix} 1 & x & x \\ 1/x & 1 & 1 \\ 1/x & 1 & 1 \end{pmatrix}$$

$$\mathrm{RGGM} \rightarrow \begin{pmatrix} (x^{n-1})^{1/n} \\ \left(\frac{1}{x}\right)^{1/n} \\ \left(\frac{1}{x}\right)^{1/n} \end{pmatrix}$$

The Generalized Balanced Scale

- Generalized Balanced Scale

$$c(x,n) = \frac{w_{\text{bal}}(x)}{1 - w_{\text{bal}}(x)}(n-1)$$

x judgment

n number of criteria

M maximum of judgment scale

- We use

$$w_{\text{bal}}(x) = w_{\text{eq}} + \left[\frac{w_{\text{max}} - w_{\text{eq}}}{M-1}\right](x-1)$$

$$w_{\text{eq}} = \frac{1}{n}$$

$$w_{\text{max}} = \frac{M}{n+M-1}$$

$$w_{\text{bal}} = \frac{1}{n} + \left[\frac{\frac{M}{n+M-1} - \frac{1}{n}}{M-1}\right](x-1)$$

The Generalized Balanced Scale

- Generalized Balanced Scale

$$c(x,n) = \frac{w_{\text{bal}}(x)}{1 - w_{\text{bal}}(x)}(n-1)$$

x judgment

n number of criteria

M maximum of judgment scale

- The generalized balanced scale can be written as

$$c(x,n) = \frac{9 + (n-1)\,x}{9 + n - \mathrm{x}}$$

- *c* resp. 1/*c* are the entry values into the decision matrix and
- *x* the pairwise comparison judgments.

The Generalized Balanced Scale

Weights for $r = c$

$$w_{\mathrm{AHP}} = \frac{r}{r+n-1}$$

AHP fundamental scale:

$$c = \mathrm{x}$$

$$w_{\mathrm{AHP}} = \frac{x}{x+n-1}$$

Balanced scale:

$$c = \frac{9+x}{11-\mathrm{x}}$$

$$w_{\mathrm{AHP}} = \frac{x+9}{(2-n)x+11n-2}$$

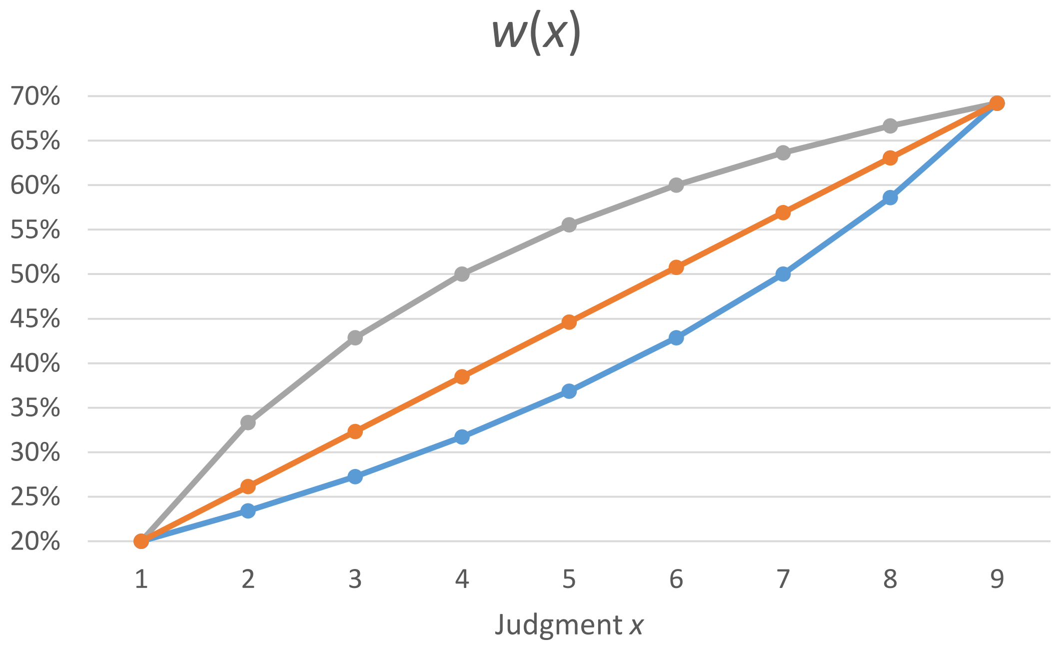
Generalized balanced scale:

$$c = \frac{9+(n-1)\,x}{9+n-\mathrm{x}}$$

$$w_{\mathrm{AHP}} = \frac{9+(n-1)x}{n(n+8)}$$

The Generalized Balanced Scale

- Example for n = 5 criteria



- AHP fundamental scale $c = x$
- Balanced scale $c = \frac{9+x}{11-x}$
- Generalized balanced scale

- For all n > 2 weights of the balanced scale are not balanced and underweighted.

The Generalized Balanced Scale

- Example

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Level 0	Level 1	Glb Prio.
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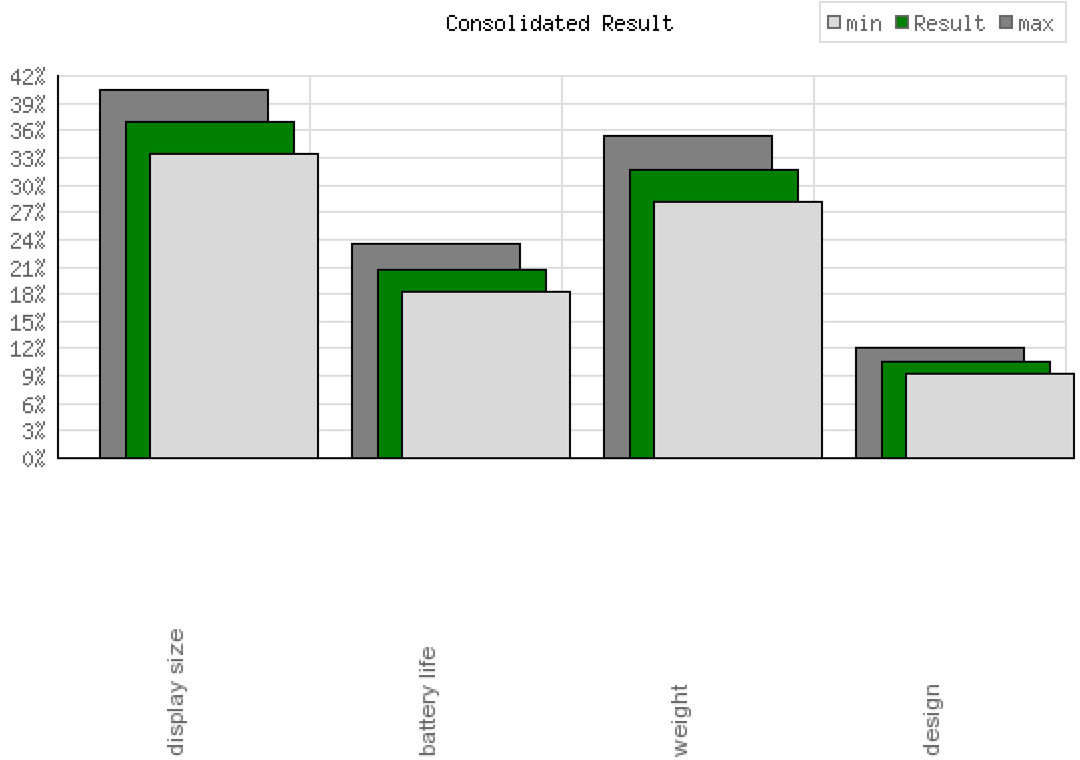
	battery life 0.163	16.3%
	weight 0.308	30.8%
	design 0.065	6.5%
		1.0

Participants	display size	battery life	weight	design	CR_{max}
Group result	**37.0%**	**20.7%**	**31.7%**	**10.6%**	**1.2%**
(+)	3.5%	2.8%	3.8%	1.4%	n/a
(-)	3.6%	2.6%	3.6%	1.2%	n/a

- Uncertainties



The Generalized Balanced Scale

Conclusion

- The so-called balanced scale has to be generalized and has to take into account the number of criteria in order to be applied for more than two criteria.
- When using the balanced scale for more than two criteria, local priorities will not be balanced and will be underweighted compared to the generalized balanced scale and the fundamental AHP scale.
- The generalized balanced scale improves weight dispersion and has lower weight uncertainties.

Goepel, K.D.,
Comparison of Judgment Scales of the Analytical Hierarchy Process - A New Approach,

Preprint of an article submitted for consideration in *International Journal of Information Technology and Decision Making* © 2017 World Scientific Publishing Company http://www.worldscientific.com/worldscinet/ijitdm (2017)

The Generalized Balanced Scale

Thank You!