# SOLVING THE LEAST SQUARES METHOD PROBLEM IN THE AHP FOR 3´3 AND 4´4 MATRICES

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## Abstract

The Analytic Hierarchy Process developed by Thomas L. Saaty (1980) is a procedure for representing the elements of any problem, hierarchically. It breaks a problem into smaller parts and then guides decision makers through a series of pairwise comparison judgments to express the relative strength or intensity of the impact of the elements in the hierarchy. These judgments are converted into numbers.
We study only one part of the decision problem, i.e. when one matrix is obtained from pairwise comparisons. Suppose that $A = [a_{ij}]$ $(i, j = 1,...,n)$ is a pairwise comparison matrix of size $n \times n$.
We want to find a positive weight vector $(w_1, w_2,..., w_n)^T \in \mathbf{R}_+^n$ representing the priorities. The Eigenvector Method *(EM)* (Saaty, 1980) and some distance minimizing methods such as the Least Squares Method *(LSM)* (Chu, Kalaba, Spingarn, 1979), Logarithmic Least Squares Method *(LLSM)* (Crawford, Williams, 1985; De Jong, 1984), Weighted Least Squares Method *(WLSM)* (Chu, Kalaba, Spingarn, 1979) and Chi Squares Method *($X^2M$)* (Jensen, 1983) are of the tools for computing the priorities of the alternatives. In the paper we study the Least Squares Method (LSM) which is a minimization problem of the Frobenius norm of $\left( A - (w_1, w_2,..., w_n)\left(\frac{1}{w_1}, \frac{1}{w_2},..., \frac{1}{w_n}\right)^T \right)$.

**Least Squares Method *(LSM)***

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left( a_{ij} - \frac{w_i}{w_j} \right)^2$$

$$\sum_{i=1}^{n} w_i = 1,$$

$$w_i > 0, \quad i = 1,2,...,n.$$

*LSM* is rather difficult to solve because the objective function is nonlinear and usually nonconvex, moreover, no unique solution exists (Jensen, 1983, 1984) and the solutions are not easily computable.
*LSM* problem for 3×3 matrices was examined by Bozóki (2003). He observed when the partial derivatives of the objective function become zero and got a polynomial system of two equations and two unknowns. This system can be solved by using *resultant* (Kurosh, 1971). Resultant of two polynomials $f$ and $g$ becomes zero if and only if there exists a common root of $f$ and $g$. The problem of finding common roots of two polynomials of two variables was reduced to finding the positive real roots of a polynomial of one variable.

In the case of $4 \times 4$ matrices, the minimization problem leads to a system of 3 polynomials of three variables. Theory of resultants was extended by Bezout (White, 1909) and Dixon (1908). Kapur, Saxena and Yang (1994) showed a method which makes it possible to solve larger polynomial systems in practice. Co-author Lewis constructed the computer algebra system FERMAT for polynomial and matrix problems. Implementing the Kapur-Saxena-Yang method, the system of 3 polynomials of 3 variables can be solved.

Numerical examples show that *LSM* solution may be the same as *EM* solution (in the case of consistent matrices), may be very close to each other. *LSM* may have non-unique solutions and even if one is unique, it may differ from the *EM* solution very much. We note that an *LSM* objective function generates a measure of inconsistency, which is different from the inconsistency by Saaty.

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