A General Principle of Rank Preservation for Adding Elements

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ABSTRACT

In this paper, we discuss a general principle of rank preservation when a new group of elements are added. We begin with a basic concept of cross comparison matrix. Then a necessary and sufficient condition releted to rank preservation is presented and proven. In addintion, we discuss a extended result about the cross comparison matrix.

The elements in a level of a hierarchy may increase or decrease when the environment of decision making changs. The priority of the elements will be affected by added a new group of elements. We hope that the priority of old elements under a single criterion does not change when the new group of elements are added. Xu [1] discussed the case when a new element is added under a single criterion and gave a necessary and sufficient condition of priority preservation in that case. For practical considerations we often meet that a group of new elements are added or group of old elements are deleted. The question now is " What happen to priority in this case ?" In this paper, we discuss a necessary and sufficient condition under which the priorities of old elements and of new elements can be preserved. We call the condition as one of a general principle of priority preservation.

Suppose that a group of old elements is $A:\{A_1, A_2, \ldots, A_n\}$, $n>=1$, the matrix of pairwise comparisons is denoted as

$$A=(a_{ij}), \quad a_{ij}=1/a_{ji}, \qquad (i,j=1,\ldots,n)$$

We will consider the elements group and their matrix of pairwise comparison as the same A.

Let a group of added elements to be $B:\{B_1, \ldots, B_n\}$, $m>=1$. Their matrix of pairwise comparisons is also denoted by

$$B=(b_{ij}), \quad b_{ij}=1/b_{ji}, \quad (i,j=1,\ldots,m)$$

Then we denote the new group of elements consisting of A and B as

$$A^* = \{ A_1, \ldots, A_n, B_1, \ldots, B_m \}$$

A^* is called the composition of A and B, A and B is called a component of A^*; respectively. The pairwise comparison matrix is

$$A^* = (a^*_{ij}) = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$$

where

$$a^*_{ij} = 1/a^*_{ji} \quad (i,j=1,\ldots, m+n)$$

$C=(c_{ij})$ $(i=1,\ldots,n, j=1,\ldots,m)$ is the matrix of pairwise comparisons of the elements in group A to the elements in group B. Similarly, $D=(d_{ij})$ is the matrix of pairwise of elements B to A. C,D are called the cross pairwise comparison matrices of A and B, B and A, respectively. Obviously,

$$c_{ij} = 1/d_{ij} > 0 \quad (i=1,\ldots,n, j=n+1,\ldots,m+n)$$

Suppose the principal eigenvector of A and B are $w=(w_1,\ldots,w_n)^T$ and $v=(v_1,\ldots,v_m)^T$ which are the priorities of elements in group A and in group B, respectively. The following theorem is important for the previous question:

Theorem 1. When a new group B is added to a group A to composite group A^*, the original priorities of the elements in group A and B are preserved, if DC and CD have the same principal eigenvalue

$$\lambda_{max}(DC) = \lambda_{max}(CD)$$

and the corresponding eigenvectors are w and v, respectively,

Proof : Necessaryty.

Assume that the priorities are preserved where A and B are conponented as A^*. In fact, there is exists a positive number k such as $W^* = [\, w^T, kv^T \,]$ is the priority of A. Therefore

$$A^* \tilde{w}^* = \lambda_{max}(A^*) w^*$$

i.e.

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix} \begin{pmatrix} w \\ kv \end{pmatrix} = \lambda_{max}(A^*) \begin{pmatrix} w \\ kv \end{pmatrix}$$

in the form of block matrix. By the matrix mulptipler we have

$$Aw + kCv = \lambda_{max}(A^*)\, w \qquad (1)$$

$$Dw + kBv = \lambda_{max}(A^*) kv \qquad (2)$$

considering that w and v are the principal eigenvectors of A and B respectivelly. We have

$$kCv = [\lambda_{max}(A^*) - \lambda_{max}(A^*)]w = \Delta\lambda_1 w \qquad (3)$$

$$Dw = \lambda_{max}(A^*) - \lambda_{max}(B) kv = \Delta\lambda_2 kv \qquad (4)$$

where $\Delta\lambda_1 = \lambda_{max}(A^*) - \lambda_{max}(A^*)$ $\Delta\lambda_2 = \lambda_{max}(A^*) - \lambda_{max}(B)$;

Multipling (3) with D , (4) with C, we obtain that

$$kDCv = \Delta\lambda_1 Dw \qquad (5)$$

$$CDw = k\Delta\lambda_2 Cv \qquad (6)$$

Substituties (4) into (5), (3) into (6), we have that

$$DCv = \Delta\lambda_1 \Delta\lambda_2 v \qquad (7)$$

$$CDw = \Delta\lambda_1 \Delta\lambda_2 w \qquad (8)$$

Since the vectors on the left side of (3),(4) are both positive one's. It follows that $\Delta\lambda_1 > 0$, $\Delta\lambda_2 > 0$. So that v and w are the principal vectors of DC and CD correspondiy to $\eta = \Delta\lambda_1 \Delta\lambda_2$.

Considering that v and w are positive, DC and CD are positive matries, we concluse that $\lambda_{max}(DC) = \lambda_{max}(CD)$ according to Perron-Frobineus.

Ufficienty: Assume. that v and w are the principal eigenvectors of DC and CD corresponding to a same eigenvalue η . We have

$$DCv = \eta v \qquad (9)$$
$$CDw = \eta w \qquad (10)$$

Multipling (9) with C, (10) with D, we obtain

$$CD.Cv = \eta Cv \qquad (11)$$
$$DC.Dw = \eta Dw \qquad (12)$$

Comparising (9) with (11), (10) with (12), we have

$$Cv = \alpha w \qquad Dw = \beta v \qquad \alpha > 0 \qquad \beta > 0 \qquad (13)$$

Substituting (13) into (9) or (10), we obtain

$$\eta v = DCv = D(\alpha w) = \alpha Dw = \beta v \tag{14}$$

i.e $\alpha\beta = \eta$

Let λ^* to be the positive maximum root of the equation

$$[\lambda^* - \lambda_{max}(A)][\lambda^* - \lambda_{max}(B)] = \eta \tag{15}$$

Since

$$= (\lambda_1 + \lambda_2)^2 - 4(\lambda_1\lambda_2 - \eta) = (\lambda_1 - \lambda_2)^2 + 4\eta > 0$$

and

$$\lambda^* = \frac{(\lambda_1 + \lambda_2) \pm \sqrt{(\lambda_1 + \lambda_2)^2 + 4\eta}}{2} \tag{16}$$

where $\lambda_1 = \lambda_{max}(A)$, $\lambda_2 = \lambda_{max}(B)$, the positive maximum root must exist. From (16), it is obvious that $\lambda^* > \max\{\lambda_1, \lambda_2\}$. Therefore $\Delta_1\lambda_1 = \lambda^* - \lambda_1 > 0$, $\Delta\lambda_2 = \lambda^* - \lambda_2 > 0$. So that $\Delta\lambda_1 \Delta\lambda_2 = \eta$.

It follows that

$$\alpha = \frac{1}{k}\Delta\lambda_1, \quad \beta = k\,\Delta\lambda_2 \tag{17}$$

Then, from (13), we obtain

$$Aw + kcv = \lambda_1 w + k\alpha w = \lambda_1 w + \Delta\lambda_1 w = \lambda^* w$$

$$Dw + kBv = \beta v + k\lambda_2 v = (\Delta\lambda_2 + \lambda_2)kv = \lambda^* v$$

We conclude that

$$A^* w^* = {}^* w^*$$

Where $w = [w^T, kv^T]$ is a positive vector. It is known that A is positive metirx. It is proved $\lambda^* = \lambda_{max}(A^*)$ and w^* is the principal eiger vector by Perron--Frobineus.i. e. w and v are the priorities of elements in A and elements in B, respectively.

Let m=1 in the above theorem. It is not difficult to obtain the theorem in [1].

We can see the importance of the cross matrix of pairwise comparison of elements in A and elements in B. But, because the pairwise comparisons is, in some degree, of subjectivity in the judgment process, therefore that makes the study of the cross matrix difficult. And the subjectivity of the judgment is the reason that makes the pairwise comparison matrix inconsistent. If the pairwise comparison matrix were consistant, our study would be easier. It is not difficult to get the following theorem.

Theorem 2. Let A be consistent. We have

(1). AC=nC BD=mD

(2). CD=nA DC=mB

adding m=n

(3). AB=(trD)C AD=(trD)A

(4). BC=(trC)B BA=(trC)D

(5). CA=(trC)A CB=nC

(6). DA=nD DB=(trD)B

(7). A•B=C•D •is Hadamard Product

From (1) and (5), (1) and (6), if m=n, we have

(8). AC=CB DA=BD

It is easy to obtain the properties in theorem subtitute

$$a_{ij} = \frac{w_{ij}}{w_j}$$

Into to the above experession.

We call matrices C and D as transforable matrix each other if there exist positive vectors $w \in R^m$, $v \in R^m$ so that $Cv = \alpha w$, $Dw = \beta v$

where C, D, are nxm, mxn positive matrices respectively. w, v are called transforable vectors of C and D, respectively.

According to that the definition, it is obvious that the matrices of cross comparison C and D are transforable and $w^* = [w^T, v^T]^T$ is their eigenvector if A is consistent matrix.

Concerning with transforable matrix, we have the following theorem 3.

Theorem 3. C and D are transforable matrices if and only of

$$\lambda_{max}(CD) = \lambda_{max}(DC)$$

Proof: Let C and D are transforable matrices. $w \in R^n$, $v \in R^m$ are eigenvectors of C and D, respectively, then

$$Cv = \alpha w \qquad Dw = \beta v \qquad (18)$$

Since the matries C, D and eigenvectors are positive we know that $\alpha > 0$ $\beta > 0$ and then we use D to multipler the first explasion of (18) we obtain that $DCv = \alpha Dw = \beta v$ (19)

It is easy to show that $\alpha\beta > 0$ as the eigenvalues of CD and DC

According to Frobineus theorem and w v's positivity it follows that

$$\lambda_{max}(DC) = \lambda_{max}(CD) = \alpha\beta \tag{20}$$

Inveresly, Let $\lambda_{max}(DC) = \lambda_{max}(D) = \eta$ (21)

We conclude that there exist positive vectors $w \; R^n$, $v \; R^m$ that

$$DCv = \eta v, \qquad CDw = \eta w \tag{22}$$

Therefore

$$CDCv = \eta Cv \tag{23}$$

It shows that Cv is eigenvector of CD respect to η .From Frobineus theorem, we know

$$Cv = \alpha w$$

In the same way,

$$Dw = \beta v$$

So, C and D are transforable matrices.

By the property of transforable matrix, we can rewrite the theorem 1 as the following:

Theorem 4. After component of A and B, the necessary and sueffecient conditions for preservation of priorities. The matrices of cross pairwise comparison C and D are transforable and w and v are transforable vectors.

REFERENCES

[1] S. Xu, 1988, The Principle of the AHP, Tianjin University Press
[2] T. L. Saaty, 1987, "Concepts, theory, and techniques: rank generation, preservation and reversal in the AHP," Decision Sciences, Vol. 18, pp. 157-177