ANALYSIS OF VARIANCE IN THE ANALYTIC HIERARCHY PROCESS

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ABSTRACT

In 1977, T. L. Saaty developed the Analytic Hierarchy Process, which is widely applied in decision sciences. For each level of the hierarchy, a pairwise comparison matrix A is made by each judge in a group and the weight vector is then derived from the matrix. Very often, these weight vectors are different from each other. This paper applies the multivariate analysis of variance to test if these weight vectors are statistically different provided each judge can repeat his or her experiment several times.

1. Introduction

In the Analytic Hierarchy Process we very often have several judges to do pairwise comparisons for any p given objects in a given level of the hierarchy and get several sets of weight vectors.

Due to sampling errors and differences among the judges, the sets of weight vectors thus derived might be different. Our concern is whether these differences in the observed sets of weight vectors are statistically significant. If they are not statistically significant, we can therefore assume that the observed differences among the sets of weights are due to sampling errors.

A way to attack this question is to use the method of multivariate analysis of variance of one way classification, which can be found in any standard multivariate textbook. We will follow in this paper the approach of Donald F. Morrison (1976) in his book, Multivariate Statistical Methods.

2. Theory and method

Suppose there are k judges and p objects to be compared in a given level of the hierarchy. Each judge represents one treatment in the analysis of variance. The jth judge, $j = 1, 2, \ldots, k$, is asked to do N_j pairwise comparisons independently. In many cases, this assumption of independence can be made true through experiment. Based on the results of the jth judge, we get N_j eigenvectors, $\vec{w}_{1j}, \ldots, \vec{w}_{N_j j}$, which are all the estimates of the true weight vector, where

$$\vec{w}_{ij}^{t} = (w_{ij1}, w_{ij2}, \cdots, w_{ijp})$$

is the eigenvector of the pairwise comparison matrix made by the jth judge in his ith set of pairwise comparisons.

Define

$$\vec{\mu}_j^{t} = \mathrm{E}(\vec{w}_{ij}^{t}) = (\mathrm{E}(w_{ij1}), \mathrm{E}(w_{ij2}), \cdots, \mathrm{E}(w_{ijp})) = (\mu_{j1}, \mu_{j2}, \ldots, \mu_{jp})$$

for $j = 1, \ldots, k$

Our linear model can then be written as

$$w_{ijl} = \mu_{jl} + \epsilon_{ijl}$$

for $j = 1, \ldots, k$, $i = 1, \ldots, N_j$, $l = 1, \ldots, p$, where ϵ_{ijl} is the error term.

The null hypothesis we are interested in testing is

$$H_0 : \vec{\mu}_1 = \vec{\mu}_2 = \cdots = \vec{\mu}_k.$$

But notice that $\sum_{l=1}^{p} w_{ijl} = 1$, which implies that $\sum_{l=1}^{p} \mu_{jl} = 1$, for $j = 1, \ldots, k$. Therefore the null hypothesis

$$H_0 : \vec{\mu}_1 = \vec{\mu}_2 = \cdots = \vec{\mu}_k$$

is equivalent to the null hypothesis

$$H_0 : \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{1,p-1} \end{pmatrix} = \begin{pmatrix} \mu_{21} \\ \mu_{22} \\ \vdots \\ \mu_{2,p-1} \end{pmatrix} = \cdots = \begin{pmatrix} \mu_{k1} \\ \mu_{k2} \\ \\ \mu_{k,p-1} \end{pmatrix}$$

Thus we would drop the last component from all the eigenvectors $\vec{w}_{ij}$ to test H_0.

Define

$$\vec{y}_{ij}^{t} = (w_{ij1}, w_{ij2}, \cdots, w_{ij,p-1}),$$

$$\vec{\epsilon}_{ij}^{t} = (\epsilon_{ij1}, \epsilon_{ij2}, \ldots, \epsilon_{ij,p-1}),$$

$$Y = \begin{pmatrix} \vec{y}_{11}^{t} \\ \vec{y}_{21}^{t} \\ \vdots \\ \vec{y}_{N_1 1}^{t} \\ \vdots \\ \vec{y}_{1k}^{t} \\ \vec{y}_{2k}^{t} \\ \vdots \\ \vec{y}_{N_k k}^{t} \end{pmatrix},$$

$$\mathrm{E} = \begin{pmatrix} \vec{\epsilon}_{11}^{t} \\ \vec{\epsilon}_{21}^{t} \\ \vdots \\ \vec{\epsilon}_{N1}^{t} \\ \vdots \\ \vec{\epsilon}_{1k}^{t} \\ \vec{\epsilon}_{2k}^{t} \\ \vdots \\ \vec{\epsilon}_{N_k k}^{t} \end{pmatrix},$$

$$X = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 1 \\ & & & & \vdots \\ 0 & 0 & & 0 & 1 \end{pmatrix},$$

where X has been partitioned into k $N_j \times k$ submatrices,

$$\mathbf{B} = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1,p-1} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k1} & \mu_{k2} & \cdots & \mu_{k,p-1} \end{pmatrix},$$

and

$$N = N_1 + N_2 + \cdots + N_k.$$

Then our linear model can be written as

$$Y = X\mathbf{B} + \mathbf{E}.$$

To apply the method of analysis of variance, we have to make the distribution assumption on the $\bar{y}_{ij}$'s, namely, the $\bar{y}_{ij}$'s have the independent normal distribution with mean vector $(\mu_{j1}, \mu_{j2}, \ldots, \mu_{j,p-1})^t$ and common unknown covariance matrix Σ for $j = 1, \ldots, k$, $i = 1, \ldots, N_j$.

To test the null hypothesis

$$H_0 : \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{1,p-1} \end{pmatrix} = \begin{pmatrix} \mu_{21} \\ \mu_{22} \\ \vdots \\ \mu_{2,p-1} \end{pmatrix} = \cdots = \begin{pmatrix} \mu_{k1} \\ \mu_{k2} \\ \vdots \\ \mu_{k,p-1} \end{pmatrix}$$

against the alternative hypothesis H_A that at least two of the above mean vectors are different, we first compute the following two $(p-1) \times (p-1)$ matrices H and E. The (rs)th element of H is defined to be

$$h_{rs} = \sum_{j=1}^{k} \frac{T_{jr} T_{js}}{N_j} - \frac{1}{N} G_r G_s.$$

The (rs)th element of E is defined to be

$$e_{rs} = \sum_{j=1}^{k} \sum_{i=1}^{N_j} w_{ijr} w_{ijs} - \sum_{j=1}^{k} \frac{T_{jr} T_{js}}{N_j}$$

where

$$T_{jr} = \sum_{i=1}^{N_j} w_{ijr},$$

and

$$G_r = \sum_{j=1}^{k} T_{jr}$$

for $r = 1, 2, \ldots, p-1$, $s = 1, 2, \ldots, p-1$.

The H matrix gives us the sum of squares due to the differences among the judges and the E matrix gives us the sum of squares due to the sampling errors.

We then calculate the greatest eigenvalue c_s of matrix HE^{-1}. Define $\theta_s = \frac{c_s}{1+c_s}$, with parameters $s = min(k-1, p-1)$, $m = \frac{|k-p|-1}{2}$, and $n = \frac{N-k-p}{2}$. In Morrison's book there is a distribution function table for the random variable θ_s with the above parameters. The decision rule is as follows: If $\theta_s \leq c_\alpha^*$, accept

H_0; if $\theta_s > c_\alpha^*$, reject H_0, where c_α^* is the critical value found from the distribution table of θ_s with the approriate parameters s, m, and n and with the given significance level α.

If H_0 is rejected at α level of significance, we then would use the methods of multiple comparisons to get the $100(1-\alpha)$ percent simultaneous confidence intervals on all linear functions of the means contrast as follows:

$$\sum_{h=1}^{p-1}\sum_{j=1}^{k} a_h c_j \bar{w}_{jh} - \sqrt{\frac{c_\alpha^*}{1-c_\alpha^*}\vec{a}^t E\vec{a}\left(\sum_{j=1}^{k}\frac{c_j^2}{N_j}\right)}$$
$$\leq \sum_{h=1}^{p-1}\sum_{j=1}^{k} a_h c_j \mu_{jh}$$
$$\leq \sum_{h=1}^{p-1}\sum_{j=1}^{k} a_h c_j \bar{w}_{jh} + \sqrt{\frac{c_\alpha^*}{1-c_\alpha^*}\vec{a}^t E\vec{a}\left(\sum_{j=1}^{k}\frac{c_j^2}{N_j}\right)},$$

where $\bar{w}_{jh} = \sum_{i=1}^{N_j}\frac{w_{ijh}}{N_j}$, $\vec{a}^t = (a_1, a_2, \ldots, a_{p-1})$ is any $p-1$ dimension vector, $\sum_{j=1}^{k} c_j = 0$ and c_α^* is the critical value of the distribution table of θ_s with parameters $s = min(k-1, p-1)$, $m = \frac{|k-p|-1}{2}$, and $n = \frac{N-k-p}{2}$.

In particular, if $\vec{c}^t = (c_1, c_2, \ldots, c_k)$ is the vector with one in the jth position and negative one in the (j+1)th position and zero elsewhere, we would have the $100(1-\alpha)$ percent simultaneous confidence interval for the linear compound

$$\sum_{h=1}^{p-1} a_h(\mu_{jh} - \mu_{j+1,h})$$

of the difference of the effects of judges j and j+1.

3. Conclusion

This paper applied the multivariate analysis of variance to test if the weight vectors derived from several judges are statistically different. Multivariate one way analysis of variance seems useful in handling the variations due to the sampling errors and the differences among the judges. But it requires that each judge do repeated pairwise comparisons over the objects independently, and hence the process takes more time. The assumption of multinormal distribution of the estimated weights seems to be questionable. When we have a balanced design, the test will be more powerful.

REFERENCES

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