A NOTE ON MULTIPLICATIVE OPERATIONS IN THE ANALYTIC HIERARCHY PROCESS

Thomas L. Saaty
University of Pittsburgh

ABSTRACT

This note deals with the idea that there are many ways of deriving composite priorities in the AHP that do not all lead to the same outcome. The current approach of distributing the weight of an element in a hierarchy in proportion to the priorities of the elements compared with respect to it, has reasonable intuitive justification. Raising the priorities to a power equal to the weight of the element is another method that is briefly illustrated here. Comments and references are given on the indifference, conservatism, and daring of different methods.

* * *

In the Analytic Hierarchy Process (AHP), paired comparisons are used to derive priorities for the elements in a level of a hierarchy (e.g., alternatives) in terms of each of the elements in the adjacent upper level (e.g., criteria). The resulting normalized scales are each multiplied by the priority of the element with respect to which the comparison is made and then summed for each alternative to obtain its overall priority.

This approach to synthesize priorities has intuitive appeal and is easily understood because it divides or apportions a unit, assigned to the top element or goal of the hierarchy, to the elements in the level immediately below according to priority, and each of these is in turn apportioned to the elements which it governs according to their priority and so on. Summation yields for each element its share of the unit value assigned to the goal. Justification for this arithmetic approach rests with the use of a hierarchy and of homogeneous elements in the comparisons.

There is an alternative way of synthesizing priorities. Instead of multiplying the normalized scales by the priority of their governing criterion, each value is raised to a power equal to the priority of that criterion; and instead of adding the results, they are multiplied and then normalized by dividing each by their sum.

The additive and multiplicative approaches can be related through the logarithmic law of Weber-Fechner. However, that law is assumed to apply in general and not just to homogeneous elements arranged in a hierarchic structure. Because of limitations on the ability of the mind to compare widely disparate elements, accuracy may be lost in this process.

In the absolute mode of measurement in the AHP, the intensity ratings are raised to the power of their criterion and in the end when alternatives are ranked according to these intensities, the corresponding intensities are multiplied rather than summed.

Which arithmetic to use seems to depend on the strength of stimuli. The additive or multiplicative approaches are used depending on whether the responses are a result of logical operations or subjective judgments, or take place in the senses as a result of physical stimuli.

SCHOOL SELECTION EXAMPLE

We shall illustrate the two approaches with a well known example. Three highschools, A, B, C, were analyzed from the standpoint of the author's son according to their desirability. Six dependent characteristics were selected for the comparison -- learning, friends, school life, vocational training, college preparation, and music classes. The pairwise matrices were as shown in Table 1.

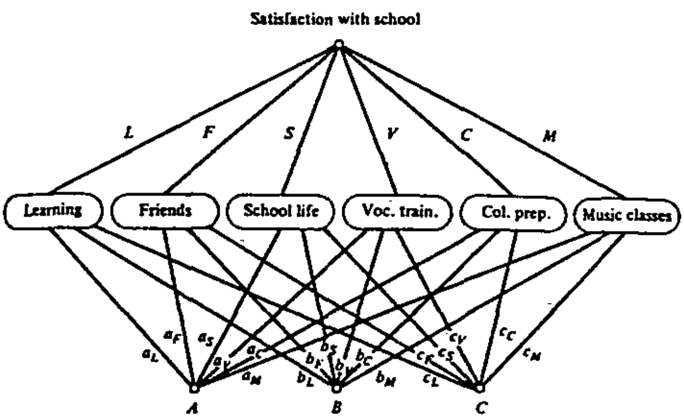

Figure 1: School satisfaction hierarchy

Table 1: Comparison of characteristics with respect to overall satisfaction with school

	Learning	Friends	School life	Vocational training	College preparation	Music classes
Learning	1	4	3	1	3	4
Friends	1/4	1	7	3	1/5	1
School life	1/3	1/7	1	1/5	1/5	1/6
Vocational training	1	1/3	5	1	1	1/3
College preparation	1/3	5	5	1	1	3
Music classes	1/4	1	6	3	1/3	1

$\lambda_{max} = 7.49$, C.I. = 0.30, C.R. = 0.24

Table 2: Comparison of schools with respect to the six characteristics

Learning	A	B	C
A	1	1/3	1/2
B	3	1	3
C	2	1/3	1

$\lambda_{max} = 3.05$
C.I. = 0.025
C.R. = 0.04

Friends	A	B	C
A	1	1	1
B	1	1	1
C	1	1	1

$\lambda_{max} = 3.00$
C.I. = 0
C.R. = 0

School life	A	B	C
A	1	5	1
B	1/5	1	1/5
C	1	5	1

$\lambda_{max} = 3.00$
C.I. = 0
C.R. = 0

Vocational training	A	B	C
A	1	9	7
B	1/9	1	1/5
C	1/7	5	1

$\lambda_{max} = 3.21$
C.I. = 0.105
C.R. = 0.18

College preparation	A	B	C
A	1	1/2	1
B	2	1	2
C	1	1/2	1

$\lambda_{max} = 3.0$
C.I. = 0
C.R. = 0

Music classes	A	B	C
A	1	6	4
B	1/6	1	1/3
C	1/4	3	1

$\lambda_{max} = 3.05$
C.I. = 0.025
C.R. = 0.04

The priority vector of the first matrix is given by

(0.32, 0.14, 0.03, 0.13, 0.24, 0.14)

THE ADDITIVE APPROACH

To obtain the overall ranking of the schools, we multiply the last matrix on the right by the transpose (column version) of the row vector of weights of the characteristics. This is the same as weighting each of the above six eigenvectors by the priority of its characteristic.

Table 3: The priorities of the schools with respect to each characteristic

	Learning	Friends	School life	Vocational training	College preparation	Music classes
A	0.16	0.33	0.45	0.77	0.25	0.69
B	0.59	0.33	0.09	0.05	0.50	0.09
C	0.25	0.33	0.46	0.17	0.25	0.22

corresponding characteristic and then adding made possible by the independence of the characteristics (see below for further elaboration). This yields:

A = 0.37
B = 0.38
C = 0.25

The son went to school A because it had almost the same rank as school B, yet school B was a private school charging close to $1,600 a year and school A was free. This was a conflict problem between the author's son and wife; the first preferred school A, and the second school B, but neither took money into consideration as important. Although the C.R. for the second level was high, they took the decision anyway despite protestations from the author about high consistency.

THE MULTIPLICATIVE APPROACH

If we raise the priority of the elements in each column to the power of the priority of the corresponding characterisic and multiply the results, we obtain Table 4 and the columns to its right:

Table 4: The priorities of the schools raised to the power of the corresponding characteristics

						Product	Additive Normalization
.556	.857	.976	.967	.717	.949	.306	.35
.845	.857	.930	.677	.847	.714	.326	.37
.642	.857	.977	.974	.717	.809	.248	.28

The two approaches lead to the same decision, although the resulting priorities are somewhat but not markedly different.

A. Easton [1] has shown that there is an infinity of rules for combining scores of alternatives and they lead to different rankings. Some of these rules are safe and some are daring.

Alternatives that receive the same score are equally meritorious. For every choice rule there is a corresponding mathematical expression which is the generating function for an indifference map. (The map would have one curve for each discrete value of the figure-of-merit.) For example, if the rule calls for computing the weighted arithmetic mean of criteria scores for an alternative, the underlying indifference map (for criteria pairs is a set of negatively sloped straight lines. (The slope of the lines depends on the criteria weights.) If the rule calls for the weighted geometric means, the underlying indifference map is a set of hyperbolas. If the rule calls for the weighted quadratic mean of scores, the map consists of a set of quarter-arcs of circles or ellipses, depending on the relative criteria weights.

According to Easton, the degree and curvature of the underlying indifference curves provides the critical clue to the inner logics of the various rules. The greater the curvature, the greater the "safety" or "daring" of the rule for alternatives with a high degree of profile scatter in their criterion score-sets. Rules with indifference curves which bend away from the origin are themost conservative; rules with curves that bend toward the origin, the most "daring." Thus the arithmetic mean is indifferent and the geometric mean is conservative.

REFERENCES

Easton, A., 1973 "One-of-a-Kind Decisions Involving Weighted Multiple Objectives and Disparate Alternatives," Paper. Multiple Criteria Decision Making, University of South Carolina, South Carolina Press, Columbia, South Carolina.

Saaty, T. L., 1988, "Decision Making: The Analytic Hierarchy Process. Planning, Priority Setting, Resource Allocation."

Vargas, L. G., 1982, "Prediction and The Analytic Hierarchy Process," Chapter. 10th IMACS World Congress on System Simulation and Scientific Computation, Montreal, Canada.