RECENT DEVELOPMENTS IN THE ANALYTIC HIERARCHY PROCESS

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1. INTRODUCTION

The Analytic Hierarchy Process(AHP), a process of measurement within hierarchic and network structures, has received great deal of attention in the past few years as a useful tool in decision making and in planning. In particular, the last year has been rich and fruitful in both theoretical and practical contributions around the world by nearly 60 authors of about 70 papers in English. Two special issues have been given to the subject, one is Vol. 20, No. 6 of Socio-Economic Planning Sciences, edited by P. T. Harker and another, Mathematical Modelling edited by L. G. Vargas and R. W. Saaty, collected 13 and 25 papers, respectively. The list of papers is found in the references.

Two papers, written by Liu Bao and Xu Shubo, R. G. Vachnadze and N. I. Mardozashvili, gave applications of the AHP and its developments in China and in the Soviet Union, respectively.

In her comprehensive survey article on the AHP, Zahedi [1986a] provided up-to-date references on the literature. This was followed by Xu's work [1986b] with an even more complete list of references including a large number of contributions made in China.

In this paper, we review recent developments of the theory of the AHP based on many papers completed in the last year. The paper consists of five parts: General theory, Hierarchic structures, Judgments, Methodology of priority estimation and General developments in the AHP.

2.GENERAL THEORY

PRINCIPLES OF THE AHP

When an individual expresses preferences among several criteria and among alternatives with respect to each criterion and then obtains an overall ranking for the alternatives using the weights of the criteria, how can he be sure that the final rank correctly reflects the strength of his preferences? Can this rank change in general if new alternatives are introduced and when might it change, and is this change legitimate? Misunderstanding the question may lead to incorrect judgments. To address these questions T. L. Saaty's paper "Concepts, theory, and techniques: rank generation, preservation and reversal in the

Analytic Hierarchy Process" introduced the ideas of absolute and relative measurement and of functional and structural dependence of criteria on alternatives when performing relative measurement. According to systems theory, the functional dependence is generally understood to be a criterion which can be used to describe behavior or change in a system. Here, structural dependence is determined by the number and arrangement of the parts to perform a function. The relative importance of the elements in performing various functions may be affected by additional structural information that is available. In the AHP, the methodology using relative comparisons and normalization mandates that structure should be considered along with function in developing the priorities. In that paper, the author represents the effects of structural transformations on the weights of the alternatives in terms of products of diagonal matrices multiplying A on the right in the following manner:

$$A C_1 C_2$$

where the jth column of the matrix $A = (a_{ij})$ is the priority of the alternatives with respect the jth criterion, the jth elements of the two diagonal matrices C_1 and C_2 are respectively $1/\sum_{i=1}^{n} a_{ij}$ and r_j/N, where $N = \sum_{j=1}^{n} r_j$ and r_j is the number of the alternatives related to the jth criterion. We represent the normalization of the priorities of the alternatives by C_j and the adjustment of the weights according to the number of alternatives by C_2.

Concerning rank reversal, the author points out that if a new alternative is added or an old one deleted, it is to be expected that the composite ranking of the other alternatives under the several criteria may change. The explanation of such rank reversal rests with the structural dependence of the criteria on the alternatives arising from the change in the number of elements and the measurement of the new alternatives both captured in the normalization operation. It is not unlike introducing an additional criteria whose importance changes each time a new alternative is added or old one deleted. There is no rank reversal with absolute measurment, which is only used when standards are established to meet the demand of prior experience. [Saaty, 1987a]

Saaty and Takizawa [1986] in conformity with the axioms of the AHP, discuss and illustrate two types of functional dependence: between sets and within a set. The former is called outer dependence of one set on another if a fundamental scale can be derived for the elements of the first set in terms of each element of the second. The latter is called inner dependence where the elements of a set are on the one hand outer dependent on a second set, and on the other conditionally dependent among themselves with respect to the elements of the second set which serve as attributes (as in input-output analysis). They note that there is no structural dependence when absolute comparisons or scoring are used because neither involves the construction of a derived scale from a fundamental one. Hence there has been little concern with structrural dependence outside the AHP and functional dependence has been the only type recognized in the literature so far.

W. A. Simpson [1986] discussed problems of a statistical nature that require investigation. In his report of 217 pages, four issues are addressed: (1) to assess the accuracy of the AHP in capturing reality, (2) to ascertain the most appropriate measuring scale for recording the pairwise comparisons between elements, (3) to determine whether the consistency ratio is a valid indicator of the likely accuracy of a respondent's recorded judgments, and, if so, then to establish whether 0.10 is the appropriate cut-off point, and (4) to ascertain the sensitivity of the AHP when answers are correct in their rank order but vary in the order of the magnitude used.

He based his research on data of subjects estimating the length of lines and the heights of people. He concluded that the AHP is a valid measuring system. Although not significantly proved, the 1-9 scale of Saaty appears to be superior to a 1-7 scale and a graphic continuum. However, he pointed out that this area requires further research in order to test other scales. He considered that the consistency ratio is a useful guide as to the likely accuracy of a respondent's answer and suggested more extensive tests. According to the results of his simulation exercise, he concludes that the AHP is a "remarkably robust measuring system".

AXIOMATIC FOUNDATION OF THE AHP

A paper concerning the axiomatic foundation of the AHP [T. L. Saaty, 1986a] appeared in the last year giving greater attention to the mathematical foundations of the AHP.

Saaty sets forth primitive notions on which the axioms are based; they are: (1) attributes or properties: A is a finite set of n elments called alternatives and C is the set of properties or attributes with respect to which the elements of A are compared; (2) Binary relation: when two objects are compared according to a property, we say that one is performing binary comparisons. The binary relation > represents "more preferred than" according to a property C. The binary relation represents "indifferent to" according to the property C; (3) fundamental scale: let P denote the set of mappings from AxA to R , $f:C \rightarrow P$, and $P = f(C)$ for

$C \in C$. Thus, every pair $(A_i, A_j) \in AxA$ can be assigned a positive real number $P(A_i, A_j) = a_{ij}$ that represents the relative intensity with which an individual perceives a property $C \in C$ in an element $A_i \in A$ in relation to other $A_j \in A$:

$$A_i >_c A_j \text{ if and only if } P(A_i, A_j) > 1$$

$$A_i \sim_c A_j \text{ if and only if } P(A_i, A_j) = 1.$$

Using these primitive notions, the author has offered the following four axioms on which the AHP is based:

AXIOM 1 (THE RECIPROCAL CONDITION)

Given any two alternatives (A_i , A_j) AxA, the intensity of preference of A_i over A_j is inversely related to the intensity of preference of A_i over A_j

$$P(A_i, A_j) = 1/ P(A_j, A_i), \quad A_i, A_j \in A, \ C \in C$$

DEFINITION 2.1 (HIERARCHY)

A hierarchy H is a partially ordered set with largest element b which satisfies the conditions:

(1) There exists a partition of H into levels $\{L_k, k = 1,2,...,h\}$, $L_1 = \{b\}$.

(2) If x is an element of the kth level ($x \in L_k$), then the set of elements "below" x where $\bar{x} = \{y \mid x \text{ covers } y\}$, k = 1,2,...,h-1, is a subset of the (k+1)st level.

(3) If x is an element of the kth level, then the set of elements "above" x ($x^+ = \{y \mid y \text{ covers } x\}$, k = 2,3,...,h, is a subset of the (k-1)st level.

DEFINITION 2.2 (HOMOGENEOUS)

Given a positive real number $\rho > 1$, a nonempty set $\bar{x} \subseteq L_{k+1}$ is said to be ρ-homogeneous with respect to $x \in L_k$ if

$$1/\rho < P_x(y_i, y_j) < \rho, \text{ for all } y_i, y_j \in \bar{x}$$

AXIOM 2 (ρ-HOMOGENEITY)

Given a hierarchy H, $x \in H$ and $x \in L_k \subset H$, then $\bar{x} \subseteq L_{k+1}$ is ρ-homogeneous for all k = 1,2,...,h-1.

DEFINITION 2.3 (OUTER DEPENDENT)

A set A is said to be outer dependent on a set C if a fundamental scale can be defined a A with respect to every $C \in C$.

DEFINITION 2.4 (INNER DEPENDENT)

Let A be outer dependent on C. The elements in A are said to be inner dependent (independent) with respect to $c \in C$ if for some (all) $A \in A$, A is outer dependent (independent) on A.

AXIOM 3 (DEPENDENCE)

Let H be a hierarchy with levels, L_1, L_2, ..., L_h. For each L_k, $k = 1,2,...,h-1$,

(1) L_{k+1} is outer dependent on L_k.

(2) L_{k+1} is not inner dependent with respect to all x L_k.

(3) L_k is not outer dependent on L_{k+1}.

DEFINITION 2.5 (EXPECTATIONS)

Expectations are beliefs about the rank of alternatives derived from prior knowledge.

AXIOM 4 (EXPECTATIONS)

All criteria and alternatives are represented in the hierarchy; i.e.

$$C \subsetneq H - L_h$$

$$A = L_h$$

THE AHP AND UTILITY THEORY

Criticisms of Utility Theory either focus on its axioms or on the construction of utility functions. L. G. Vargas [1986] developed a method for the construction of ratio scale value functions using the AHP. The method avoids the problems of uniqueness encountered in the construction of utility functions when using either the certainty equivalence or the probability equivalence methods. He showed that under the assumption of cardinal consistency, utility functions are a particular case of ratio scales. The method based on reciprocal pairwise comparisons allows decision makers to relax the transitivity assumption and help to derive a unique scaling of preference. With regard to the construction of a utility function from reciprocal pairwise comparisons, Vargas proved that a utility function of empirical objects A in a finite or infinite countable set can be expressed in the following way:

$$u(A_i) = (w_i - w_*)/(w^* - w_*) \qquad i=1,2,....,n$$

where $w = (w_1, w_2, ..., w_n)^T$ is the principal right eigenvector of the matrix M of pairwise comparisons, i.e.

$$M w = \lambda_{max} w$$

and $w^* = \max \{ w_i \}$, $w_* = \min \{ w_i \}$.

In another paper, Vargas [1987] points out that the AHP goes several steps further than utility theory in the following ways. First, it deals with pairwise comparisons providing a method to elicit judgments of individuals and synthesizing them into priorities that represent the relative attractiveness of the consequences according to criteria. Second, it is a group decision making methodology. Judgments of individuals can be all fused into a single judgment through compromises or thorugh the geometric mean criterion. Third, it can deal with several levels of complexity. Fourth, it is a true measurement theory in the sense that when there are existing absolute scales associated with the consequences, the AHP can reproduce known results. On the other hand, utility theories can only be used for individual decision makers and cannot be used to estimate numerical values from existing scales. Also, they cannot deal with more than two levels of complexity.

3. HIERARCHIC STRUCTURE

HIERARCHY

Zahedi [1986] pointed out that setting up a decision hierarchy is the most important aspect of the AHP. Saaty [1987a] writes that a hierarchy is a structure used to represent the simplest type of functional (contextual or semantic) dependence of one level or component of a system on another in a sequential manner. It is also a convenient way to decompose a complex problem in search of cause-effect explanations in steps which form a linear chain. One result of this approach is to assume the functional independence of an upper part, component, or cluster from its lower parts. This often does not imply its structural independence from the lower parts which involves information on the number of elements, their measurements, etc.

It is necessary to make a distinction between a hierarchy and a tree structure. Every tree is an incomplete hierarchy but not every hierarchy is a tree. A tree is formed from causal relations whereas a hierarchic structure need not have a simple causal origin and hence is essentially non-causal in the classical sense.

FEEDBACK SYSTEMS

Saaty [1987] had developed an even more general way than a hierarchy to structure a complex problem involving functional dependence. It allows for feedback between components. It is a network system of which a hierarchy is a special case. In both hierarchies and networks the elements within each component may also be dependent on each other. A nonlinear network can be used to identify relationships among components using one's own thoughts, relatively free of rules. It is especially suited for modelling dependence relations. Such a network approach makes it possible to represent and analyze interactions and also to synthesize their mutual effects by a single logical procedure. For emphasis we note again that in the nonlinear network or system with feedback , there are two kinds of dependence: that between components, but in a way to allow for feedback circuits, and the other within a component combined with feedback between components. He calls these respectively outer and inner dependence. If the criteria cannot be compared with respect to an overall objective because of lack of clarity of purpose, they can be compared in terms of the alternatives. Thus in such a setting the systems approach can replace the hierarchic approach.

Hamalainen and Seppalainen [1986] applied the above network or supermatrix technique, called the Analytic Network Process (ANP), to energy policy planning. They proposed the use of the ANP as a new direction in decision analysis research. It is based on the idea that human thinking and decision making are not "linear" or "hierarchical" but network like.

4. THE JUDGMENT MATRIX

RECIPROCAL MATRICES

The study of reciprocal matrices plays an important role in the development of the theory of the AHP. E. Barbeau [1987] proved that the following statements about a reciprocal matrix A of order 4 are equivalent:

(a) G is a column eigenvector of A:

(b) H is a row eigenvector of A, where the kth entry of H is the reciprocal of the kth entry of G:

(c) There exists a column vector C and a row vector R for which all entries are positive, each entry of C is the reciprocal of the corresponding entry of R, and C and R are respectively, right and left eigenvectors.

THE SCALE

Saaty [1987] has pointed that when the elements being compared are closer together than indicated by the scale, one can use the scale 1.1, 1.2,..., 1.9. If still finer, one can use the appropriate percentage refinement and so on.

Saaty and Vargas [1987a, b, c] developed a theory for constructing response scales based on the reciprocal property of paired comparisons of stimuli from the same sensory continuum. Reciprocal paired comparisons define the pair estimator function K(s,t) which is the kernel of a Fredholm integral equation of the second kind:

$$\int_{S \subset R} K(s\ t)\ w(t)\ dt = w(s)$$

where R is the set of positive real numbers. Kernels can be considered as a generalization of reciprocal matrices which arise in the process of making decisions from paired comparisons and are well suited for constructing response scales to stimuli from sensory continuua. They proved that these scales take the form of linear combinations of the dense functions $s^{\alpha} e^{-s\beta}$.

ABSOLUTE AND RELATIVE MEASUREMENT

Saaty [1986e] has advanced a theory to differentiate between the two kinds of comparisons, absolute and relative. In absolute comparisons alternatives are compared with a standard in one's memory that has been developed through experience; in relative comparisons alternatives are compared in pairs according to a common attribute. The AHP has been used with both types of comparisons to derive ratio scales of measurement. We call such scales absolute and relative measurement scales respectively. Relative measurement in the AHP is well developed. Absolute measurement (sometimes called scoring or ranking) is applied to rank the alternatives in terms of the criteria or else in terms of ratings (or intensities) of the criteria. After setting priorities on the criteria (or subcriteria, if there are any) pairwise comparisons are also performed on the ratings themselves to set priorities for them under each criterion. Finally, alternatives are scored by checking off their rating under each criterion and summing these ratings for all the criteria. This produces a ratio scale score for the alternative. The scores thus obtained for the alternatives can be normalized. In a separate paper Saaty [1986b] applied absolute measurement to rank 329 cities in the U.S. as to how livable they are according to nine criteria. His work used data from a book on the subject by Boyer and Savageaux. When using absolute measurement, no matter how many new alternatives are introduced, or old ones deleted, the ranks of the alternatives cannot reverse. Absolute measurement needs standards, often set by society for its convenience, and sometimes having little to do with the values and objectives of the judge making the comparisons. In completely new decision problems or in old problems where no standards have been established, we must use relative measurement by comparing alternatives in pairs to identify the most preferred ones.

CONSISTENCY OF THE JUDGMENT MATRIX

In the AHP, the usual procedure is for the judges to accumulate the results of their pairwise comparisons in a positive reciprocal matrix, and then to accept the resulting eigenvector as a summary of their judgments. Deturck [1987] gave an interactive approach to guide the judges in revising the pairwise comparison matrix toward consistency. He proved two theorems:

(1) If $A \in P$, is a positive reciprocal matrix with principal right eigenvector $w = (w_1, w_2, ..., w_n)^T$ and $D \in G_0$ is a diagonal matrix with positive diagonal entries $d_1, d_2, ..., d_n$, then I (A) =

DAD^{-1} is a positive reciprocal matrix with principal eigenvector $w' = (d_1w_1, d_2w_2, \ldots, d_nw_n)$. The principal eigenvalue is the same for both matrices.

(2) For each positive vector v, the set P_v of positive reciprocal nxn matrices with right principal eigenvector v is diffeomorphic to $R^{(n-1)(n-2)/2}$.

In Deturck's interactive approach to consistency, the judges provide an initial matrix A of pairwise comparisons for which the right eigenvector w is computed. The judges are then given the opportunity to adjust w_0, which yields a new vector $\bar{W}_0$. The resultof (1) is then used as follows: We form the diagonal matrix $D = D_{w_0 \bar{w}_0}$, and A_0 is conjugated by $D_0 A D_0^{-1}$ to form a new positive reciprocal matrix $\bar{A}$. This new matrix is an alternative to the original matrix, but before it is presented to the judges, (2) is used to make $\bar{A}_0$ "10% more consistent", and the resulting matrix A_1 is presented to the judges as an alternative to their original A_0. If this procedure is repeated indefinitely, the limit of the sequence of matrices A_0, A_1 .... would be a consistent matrix.

INCOMPLETE COMPARISON

Pairwise comparisons are fundamental in the use of the AHP. The judgments needed for a particular matrix of order n corresponding to the number of elements being compared, is n(n-1)/2 because it is reciprocal and the diagonal elements are equal to unity. Harker [1987] gave an extension of the approach which allows a decision maker to say "I don't know" and "I'm not sure" to some of the questions being asked. Harker's approach is based on the definition of a quasi-reciprocal matrix. A nonnegative nxn matrix A is quasi-reciprocal if

$$a_{ij} > 0 \text{ and } a_{ij} > 0 \text{ implies } a_{ij} = 1/a_{ji},\ \forall\ i\ j = 1,2 \ldots n$$

Let us assume that the decision maker has considered a set of n alternatives and has completed some subset of the n(n-1)/2 pairwise comparisons to form a matrix $C = (c_{ij})$ with $c_{ij} \geq 0$ and if $c_{ij} > 0$ then $c_{ij} = 1/c_{ji}$. Let $B = (b_{ij})$ be an nxn matrix formed from the partially completed matrix C as follows:

$$b_{ij} = c_{ij} \quad \text{if } c_{ij} \text{ is a positive number}$$
$$= \text{otherwise}$$
$$b_{ii} = m_i, \text{ the number of unanswered questions in row } i, \quad i, j = 1, \ldots, n$$

The matrix $A = I+B$ is primitive, i.e., there is an intager $k > 1$ such that A^k is positive. Therefore, the solution of the eigenvalue problem for A can be considered as the priority of the alternatives under incomplete comparisons. Harker also proved that the Perron root of a nonnegative, irreducible, quasi-reciprocal matrix A is greater than or equal to n, the rank of A, and is equal to n if and only if A is consistent; i.e.,

$$a_{ik} a_{kj} = a_{ij} \quad \text{for all } i,j,k \text{ with } a_{ij}, a_{ik}, a_{kj} \text{ positive.}$$

GROUP JUDGMENTS AND THEIR SYNTHESIS

When dealing with group judgments, Saaty has proposed that any rule to combine the judgments of several individuals should also satisfy the reciprocal property. A proof that the geometric mean, which makes no requirement on who should vote first, satisfies this condition was generalized in the papers by J. Aczel and C. Alsina [1986, 1987]. In the first paper they proposed that an assumption involving the following separability condition (S) be considered:

$$f(x_1, x_2, \ldots, x_n) = g(x_1)\, g(x_2) \ldots g(x_n) \quad \text{for all } x_1, x_2, \ldots, x_n$$

along with the unanimity conditions (U):

$$f(x, x, \ldots, x) = x \quad \text{for all } x \text{ in an interval of real numbers } P.$$

The authors proved that a synthesizing function $f: P^n \rightarrow P$ is separable (S) (with continuous nonconstant g and continuous, cancellative and associative) and has the unanimity property (U) if, and only if, f is of the following form:

$$f(x_1, x_2, \ldots, x_n) = g^{-1}\left((g(x_1) + \ldots + g(x_n))/n \right)$$

with an arbitrary continuous and strictly monotonic g, and thus f is a quasiarithmetic mean. Under these circumstances P must be open or half-open. Some well-known quasiarithmetic means include the arithmetic mean, the geometric mean, the harmonic mean, the root-mean-square, the root-mean-power and the exponential mean. In many situations, where ratio judgments are used, it is reasonable to assume, in addition to (S) and (U), the following reciprocal property(R):

$$f(1/x_1, 1/x_2, \ldots, 1/x_n) = 1/f(x_1, \ldots, x_n)$$

Aczel and Saaty [1983] proved that if P is an interval of positive numbers which with every element also contains its reciprocal, a synthesizing function f:P → P is separable (S), has the unanimity (U) and the reciprocal (R) properties if and only if f is of the following form:

$$f(x_1, \ldots, x_n) = \exp g^{-1} [g(\log x_1)+\ldots+g(\log x_n)]/n$$

where g is an arbitrary, continuous, strictly monotonic and odd function. If, in particular, g(x)=x, then we have

$$f(x_1, \ldots, x_n) = (x_1 x_2 \ldots x_n)^{1/n}$$

the geometric mean. In the paper of Aczel and Alsina [1986] the homogeneity and power conditions are also discussed.

5. PRIORITY

THE METHODS FOR ESTIMATING PRIORITY

In using the AHP, one important mathematical question is how to derive the priority vector from the matrix of paiwise comparisons. Many methods have been proposed to derive the priority, such as the right eigenvector (EM), the left eigenvector, the arithmetic mean of the rows, the logarithmic least squares (LLSM or the geometric row mean), the method of least squares (LSM) and so on. Saaty [1987a] has shown that when the matrix of comparisons is inconsistent, to capture inconsistency, the principal right eigenvector is the "best" way to estimate the priority vector by using the concept of dominance walks. The dominance of an alternative along all walks of length k > m is given by

$$(1/m) \sum_{k=1}^{m} (A^k e)/(e^T A^k e)$$

He proved the following theorem:

The dominance of each alternative along all walks k, as k is given by the solution of the eigenvalue problem $A w = \lambda_{max} w$

He compared the eigenvector solution and the LLSM solution with the following example giving rise to rank reversal by the two methods:

	A	B	C	D	E	Eigenvector solution	LLSM solution
A	1	1/6	1/2	1/9	5	.0893	.0819
B	6	1	2	1	5	.3287	.3433
C	2	1/2	1	1	5	.1983	.2089
D	9	1	1	1	5	.3413	.3214
E	1/5	1/5	1/5	1/5	1	.0424	.0418

In the eigenvector solution the alternatives are ranked in descending order D, B, C, A, E, whereas the LLSM solution ranks them of B, D, C, A, E.

In comparing various methods for deriving the priority, Fichtner[1986] presented an axiomatic approach to decide which method is "best". These axioms are (1) correctness in the consistent case, (2) comparison order invariance, (3) smoothness, (4) power invariance. The author proved that LLSM fulfills all these axioms. If the axiom (4) is repacked by the axiom of rank preservation, the EM fulfills these axioms.

Zahedi [1986b] addressed the accuracy and rank preservation properties of various estimation methods in recovering the true relative weights at one level of the hierarchy by a simulation procedure. The estimation methods she compared in her paper included the eigenvalue method, the mean transformation, the row geometric mean, the column geometric mean, the harmonic mean and the simple row mean. Her simulation analysis compares the methods based on measures of statistical accuracy and rank preservation. Since the probability distribution of the error term may affect the preformance of the methods, the analysis is carried out for three distributions with nonnegative random variables: gamma, lognormal and uniform distributions.

She introduced the mean transformation method estimator given by:

$$\min_{w_j > 0} \sum_{i > j} (b_{ij} - w_j)^2$$

where b_{ij} is the element of a matrix obtained from transposing A and dividing each of its row elements by the row sum. This transformation changes the elements of A from pairwise preferences to relative weights, each observed n times.

Her major results are as follow:

--The most important factors in the estimation of relative weights comprise the probability distribution of error terms and the type of input matrix.

--While analysts do not control the probability distribution of the error terms, they can improve the estimation by collecting data for the upper and lower triangles of the input matrix.

--The column geometric mean and the simple row average could be dropped from the list of estimators because they generally show the highest degree of sensitivity toward the underlying distribution of error terms, and, exhibit, in some cases, very poor accuracy and rank statistics.

--In the computation of the eigenvalue method, the "size" criterion preforms exactly as well as the "convergence" criterion, and has the additional advantage of computational efficiency, which becomes crucial in the case of a large number of elements.

--Of the four methods (excluding the column geometric mean and the simple row average), no method dominates the others in all statistics. The mean transformation method, however, is the most robust toward the underlying distribution, and the type and size of input matrix. Hence, in the absence of knowledge of the distribution of error terms, the mean transformation is recommended.

--Finally, when an alternative has a relative weight close to zero for an attribute, the symmetric type of input matrix is inappropriate because the performance of all methods deteriorates as the pairwise scores become very small or very large. The full input type does not exhibit extensive sensitivity to the extreme values, and hence constitutes the better choice.

Crawford [1987] discussed the Geometric Mean Procedure for estimating priority (i.e. LLSM) and also develop an index and related rules to judge the consistency of a matrix.

In setting priorities it is necessary to use the two sides of human experience (dominance and dominated or larger than and smaller than) to obtain a " balanced or reasonable " priority. Mathematically, the problem can be considered as a question of how to develop the matrix of dominance or the matrix of dominated, or perhaps to synthesize the left and right eigenvectors of the pairwise comparison matrix. An important question then is: what relationship is there between the left and right eigenvectors of the same reciprocal matrix? Xu [1987] has proved the following result:

The reciprocal property between corresponding components of the left and right principal eigenvectors holds if and only if

$$E \cdot e = E^T e$$

where E is the perturbation matrix in the equation A = E W, $W = (w_i/w_j)$, $e = (1,1,\ldots,1)^T$ and E/λ_{max} is a doubly-stochastic matrix.

UNCERTAINTY

There are two uncertainties in using the AHP. The first is uncertainty in the judgments, and the second is in the number of criteria and alternatives. The former uncertainty can be expressed in two ways: (1) as a point estimate with a probability distribution function, and (2) as an interval estimate without a probability distribution. Saaty and Vargas [1986] addressed the interval estimate approach by means of simulation assuming that all points of the interval are equiprobable, i.e., the simulation assumes that the random variables are uniformly distributed. Using the Kolmogorov-Smirnov test, they showed that the eigenvector components satisfy the truncated normal distribution. It is suggested that the Central Limit Theorem can be applied to the distribution of the eigenvector components as limiting averages of the dominance of each alternative over the other alternatives along paths of all lengths. They showed how alternatives are chosen according to the product of their priority and the likelihood that they do not reverse rank. This way of capturing the uncertainty of a decision maker's judgments allows one to measure jointly the importance and the likelihood of rank preservation.

When the alternatives become available to the decision maker sequentially rather than simultaneously, how does one apply the AHP? That includes the uncertainty about the value of future alternatives and also the number of alternatives. Weiss [1987] gave a technique similar to the classic "secretary problem" of operations research and described some sample results using this technique. The procedure involves prioritizing the criteria of possible alternatives before the alternatives became available, scoring the alternatives, and then comparing the score of an alternative with an easily computed (through a dynamic programming recursion) critical value.

DYNAMIC PRIORITY

When judgments change over time, the eigenvalue problem that must be solved is given by $A(t)w(t) = \lambda_{max}(t)w(t)$. The solution should be a time-dependent function. In this case we are concerned with dynamic priorities. Xu [1986] presented a new dynamic model whose pairwise comparison matrix has the following form:

$$A(t) = M(t)A_0M^{-1}(t)$$

where $M(t) = \mathrm{diag}\,(m_i(t))$ and $m_i(t_0) = 1$, $(i = 1,2,\ldots,n)$, $A_0 = A(t_0)$ is the pairwise comparison matrix at the initial time t_0. The function $m_i(t)$ represents how the ith factor's importance changes over time. Several theorems related to dynamic priorities are proved in this paper. In addition, using the dynamic priority model, Xu discussed the development of a strategy to deal with major energy resources in China until the year 2000.

6. DEVELOPMENT OF METHODOLOGY OF THE AHP

THE AHP AND OPTIMIZATION

Saaty [1986c] explored the concept of optimization by solely using the AHP and compared outcomes with those obtained in traditional optimization theory with and without constraint. The difference is essentially that using the AHP there is an absence of the traditional black box effect that involves complex manipulations in algebra or the calculus on an assumed linear or nonlinear mathematical structure. With the AHP one simply uses an individual's understanding together with a way to convert his judgments to ratios to deal with optimization. At first sight this may go contrary to one's intuition. In the end he must face the question of whether the magic of traditional manipulations gives rise to better answers than one's actual and complete understanding. Should one abdicate one's judgmental control of the solution, when and why. Saaty suggested that optimization through the AHP is now ripe for a deeper look.

Olson et. al. [1986] proposed a technique of analysis for multiple objective policy models that use the AHP as a means of initially checking for consistency. For a concave utility function over a convex feasible region, this will provide a solution that should be near the desired alternative. However, due to the expected nonlinear nature of a utility function, it will probably not be as good as possible. A simple pattern search around this initial solution will provide decision makers with alternatives for comparison. They proposed use of objective bounds as a means of controlling the search, and assuring new alternatives in the vicinity of the original alternative. Their method has the following five steps:

(1) Solve the model for each objective; obtain an optimal solution for each objective; build a payoff table to identify the worst nondominated solution for each objective, and thus the relevant ranges; present to the decision maker.

(2) Apply the AHP; check for decision maker consistency; obtain the principal eigenvector as weights for the model objective function.

(3) Solve the model using linear prooamming, obtaining the initial alternative.

(4) Generate new alternatives around the original solution; force imporvement of each objective individually by some proportion of the current attainment to the possible present alternatives to the decision maker for selection.

(5) Check for convergence: if the prior solution was selected, quit; if a new alternative is selected, return to (4), with a new selection as current solution.

Korhonen [1987] dealt with the use of the AHP for specifying a reference direction, which is used to find a search direction in the visual interactive method developed by Korhonen and Laakso for multiple criteria problems.

An important observation made by Saaty regarding rank reversal under relative comparison is that it is similar to introducing additional variables and additional equations or inequalities in an existing model. The solution of the model may be changed dramatically. Sensitivity analysis in modelling so far has said nothing about this kind of situation.

TOPICS FOR FURTHER INVESTIGATION OF THE AHP

R. Saaty [1987] presented the following topics for further investigation of the AHP:

(1) Generalization of the hierarchy and systems networks to manifolds.

(2) Deeper and more extensive research on continuous judgments.

(3) Test different group decision making approaches on the same problem and search for common elements. Develop A, B, C guidelines for group participation in decision making.

(4) Investigate further the relationship of the principal eigenvector to the Weber-Fechner Power Law.

(5) Develop applications of the AHP in Game Theory, particularly with respect to negotiation, extending Saaty's retributive conflict resolution theory.

(6) Investigate the relationship of AHP to optimization. Can the general optimization problem be solved using the AHP alone?

(7) Implement psychological studies to show how people's strength of feelings can be adequately represented by numerical scales.

(8) Study the sensitivity of priorities to the number of criteria and, more generally, to the size of the hierarchy.

(9) Sample opinions on how satisfied clients are with AHP outcomes.

(10) Formulate more cases using the AHP in resource allocation, planning, cost-benefit analysis and conflict resolution.

(11) Is there power in hierarchic formulation and judgments to make better predictions? How can it be tested?

(12) The AHP and Risk Analysis: put forth a definitive theory about the use of scenarios in risk analysis.

(13) Investigate relationship between AHP and Artificial Intelligence.

(14) Develop communication and causal languages using AHP.

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