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ON THE SOLUTION OF A SUPERHIERARCHY WITH A BAYES LOOP

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Abstract: In this paper, we discuss the uniqueness condition for the solution of a kind of a superhierarchy and establish the necessary and sufficient conditions for a superhierarchy to be a Bayes loop. Also, a simple method of finding the solution is introduced.

1. Introduction

The problem of a superhierarchy with a Bayes loop was introduced in the literature by Saaty (Saaty, 1994). This problem considers a network as shown in Figure 1, where $L_1 = \{\theta_1, \theta_2, ..., \theta_n\}$ is the state space and $L_2 = \{x_1, x_2, ..., x_m\}$ is the sample space. Let $P_1 = ((p_1)_1, (p_1)_2, ..., (p_1)_n)^T$ be the importance vector of L_1 given G_1, and $P_2 = ((p_2)_1, (p_2)_2, ..., (p_2)_m)^T$ be the importance vector of L_2 given G_2. The elements of P_1 and P_2 are all positive (the elements with zero components can be deleted). The inter-impact between L_1 and L_2 is described by two matrices P_{12} and P_{21}, respectively, where $P_{12} = ((p_{12})_{ij})$ is the column stochastic matrix whose jth column $((p_{12})_{1j}, (p_{12})_{2j}, ..., (p_{12})_{nj})^T$ is the importance vector of L_1 given x_j of L_2, and $P_{21} = ((p_{21})_{ij})$ is the column stochastic matrix whose jth column $((p_{21})_{1j}, (p_{21})_{2j}, ..., (p_{21})_{mj})^T$ is the importance of L_2 given θ_j of L_1. Naturally, P_{12}, P_{21} are both non-negative matrices.

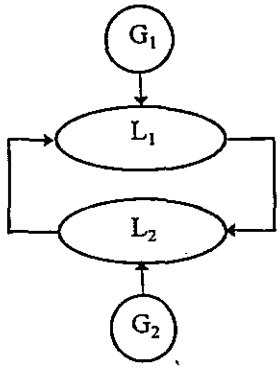

Figure 1: A Superhierarchy with a Bayes Loop

The supermatrix corresponding to the superhierarchy of Figure 1 is given by

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$$W = \begin{array}{c} \\ L_1 \\ L_2 \\ G_1 \\ G_2 \end{array} \begin{array}{c} \begin{array}{cccc} L_1 & L_2 & G_1 & G_2 \end{array} \\ \begin{pmatrix} 0 & P_{12} & P_1 & 0 \\ P_{21} & 0 & 0 & P_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

This kind of superhierarchy is used for feedback decision problems appeared in many areas. A decision making problem that doctors often face, for example, is how to diagnose the disease of a patient in determining a medical treatment plan according to the patient's symptoms. In this problem, the diseases cause some symptoms, and the symptoms reflect some diseases; that is, there exists a feedback between diseases and symptoms. The structure of these kinds of decision problems can be shown as in Figure 2.

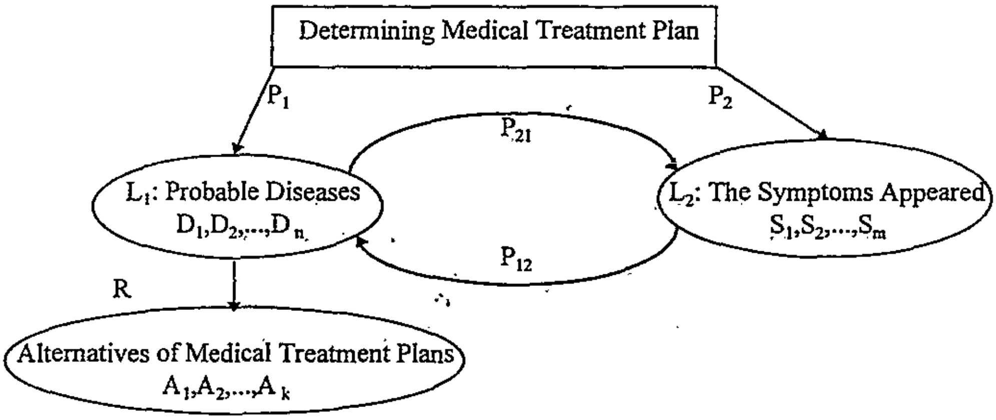

Figure 2: The Superhierarchy with a Bayes Loop for Medical Decision Problems

Here, $P_1 = ((p_1)_1, (p_1)_2,..., (p_1)_n)^T$ is the priority weights vector for the diseases of the patient for whom a medical treatment plan will be selected, and $P_2 =((p_2)_1, (p_2)_2,..., (p_2)_m)^T$ is the priority weights vector for the symptoms of the patient. The inter-impact between L_1 and L_2 is described by two matrices P_{12} and P_{21}, where the jth column of $P_{12} = ((p_{12})_{ij})_{nxm}$ is the priority weights vector of the diseases for a given symptom S_j, j = 1, 2, ..., m. Similarly, the jth column of the matrix $P_{21} = ((p_{21})_{ij})_{mxn}$ is the priority weights vector of the symptoms for a given decease D_j, j = 1, 2, ..., n. And the jth column of the matrix $R = (r_{ij})_{kxm}$ is the priority weights vector of the alternatives of medical treatment plans for a given disease D_j, j = 1, 2, ..., n.

Obviously, P_1 and P_2 are the solutions of the following equation

$$P_{12}\,b = a \quad \text{and} \quad P_{21}\,a = b \tag{1}$$

for some positive vectors $a = (a_1, a_2,..., a_n)^T$ and $b = (b_1, b_2, ..., b_m)^T$. The decision problem in medical diagnosing cases is to find P_1 and P_2 from Eq. (1) given P_{12} and P_{21} which are obtained by the doctor according to his or her medical knowledge, and then to determine the corresponding medical treatment plan. The solution of Eq. (1), however, is not always unique. Consider the following example.

Example 1: Define

$$P_{21} = \begin{pmatrix} .8 & .3 & 0 & 0 \\ .2 & .7 & 0 & 0 \\ 0 & 0 & .6 & .3 \\ 0 & 0 & .4 & .7 \end{pmatrix} \text{ and } P_{12} = \begin{pmatrix} .6400 & .1600 & 0 & 0 \\ .3600 & .8400 & 0 & 0 \\ 0 & 0 & .9474 & .8372 \\ 0 & 0 & .0526 & .1628 \end{pmatrix}$$

We can find vectors $\bar{a}, \bar{b}, \tilde{a}$ and $\tilde{b}$ as

$$\bar{a} = \begin{pmatrix} .080 \\ .120 \\ .720 \\ .080 \end{pmatrix}, \ \bar{b} = \begin{pmatrix} .100 \\ .100 \\ .456 \\ .344 \end{pmatrix} \text{ and } \tilde{a} = \begin{pmatrix} .160 \\ .240 \\ .540 \\ .060 \end{pmatrix}, \ \tilde{b} = \begin{pmatrix} .200 \\ .200 \\ .342 \\ .258 \end{pmatrix}$$

such that both satisfy Eq. (1). In fact, for any $\lambda \in [0, 1]$

$$a = \lambda \begin{pmatrix} .4 \\ .6 \\ 0 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 0 \\ .9 \\ .1 \end{pmatrix}, \ b = \lambda \begin{pmatrix} .5 \\ .5 \\ 0 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 0 \\ .57 \\ .43 \end{pmatrix}$$

constitutes a solution of Eq. (1).

Besides, in many cases, the solution of Eq. (1) also satisfy

$$(p_{12})_{ij} b_j = (p_{21})_{ji} a_i \qquad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m.$$

when we call the inter-impact between L_1 and L_2 is a Bayes loop. In the diagnostic problem of Figure 2, for example, $(p_{12})_{ij}(p_2)_j$ and $(p_{21})_{ji})(p_i)$ are the priority weights of the patient suffering from disease D_i and appearing with symptom S_j, respectively, such that

$$(p_{12})_{ij} (p_2) = (p_{21})_{ji} (p_1)_i \qquad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m.$$

In this paper, we shall first discuss the uniqueness condition of the solution of Eq. (1), then establish the necessary and sufficient conditions for the superhierarchy in Figure 1 to be a Bayes loop. We shall also describe a simple method of finding the unique vectors a and b that satisfy Eq. (1) of Bayes loop.

2. The Uniqueness Condition of the Solution of Eq. (1)

We begin with some definitions.

Definition 1: *The inter-impact between L_1 and L_2 is called a Bayes loop if*

$$(p_{12})_{ij}(p_2)_j = (p_{21})_{ji}(p_1)_i \qquad \text{for } i=1, 2, \ldots, n;\ j=1, 2, \ldots, m$$

Definition 2: *If for any fixed $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, $(p_{12})_{ij} > 0$ implies $(p_{21})_{ji} > 0$ and vice versa, we call the inter-impact between L_1 and L_2 has Property A.*

Because $(p_1)_i$ (i=1, 2, ..., n) and $(p_2)_j$ (j = 1, 2, ..., m) are all positive, from Definition 1 we see that Property A will be satisfied when the inter-impact between L_1 and L_2 is a Bayes loop.

Definition 3: *The inter-impact between L_1 and L_2 is called reducible if L_1 and L_2 can be split into two non-empty complementary subsets L_{1a}, L_{1b} and L_{2a}, L_{2b}, respectively, such that*

$$(p_{12})_{ij} = (p_{21})_{ji} = 0 \qquad \text{for } i \in L_{1a} \text{ and } j \in L_{2b}, \text{ or } i \in L_{1b} \text{ and } j \in L_{2a}$$

Otherwise, the inter-impact between L_1 and L_2 is called irreducible. We now state and prove the theorem about the uniqueness of the solution of Eq. (1).

Theorem 1: *If the inter-impact between L_1 and L_2 is irreducible with Property A, then there exist two unique positive vectors a and b that satisfy Eq. (1); that is,*

$$P_{12} b = a \quad \text{and} \quad P_{21} a = b \tag{1}$$

Proof: Eq. (1) can be rewritten as

$$\begin{pmatrix} 0 & P_{12} \\ P_{21} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2}$$

Let

$$U = \begin{pmatrix} 0 & P_{12} \\ P_{21} & 0 \end{pmatrix}$$

The inter-impact between L_1 and L_2 is irreducible implies the matrix U is irreducible. In fact, if U is reducible, then U can be partitioned (by permutation) into the form

$$\begin{pmatrix} A_1 & 0. \\ B & A_2 \end{pmatrix} \tag{3}$$

where A_1 and A_2 are square matrices. However, B in Eq.(3) must be zero because of Property A. This means the inter-impact between L_1 and L_2 is reducible, which is contrary with the condition of this theorem. So U must be irreducible. Then, we prove that

$$U^2 = \begin{pmatrix} P_{12}P_{21} & 0 \\ 0 & P_{21}P_{12} \end{pmatrix}$$

is regular, or its diagonal matrices $P_{12}P_{21}$ and $P_{21}P_{12}$ are both primitive. Let $T = (t_{ij}) = P_{12}P_{21}$. First, the diagonal elements of T must be all positive. In fact, there exists at least one positive element in the ith column of P_{21}. Let us denote this element as $(p_{21})_{hi}$. Using Property A, we have

$$t_{ii} = \sum_{k=1}^{m} (p_{12})_{ik} (p_{21})_{ki} \geq (p_{12})_{ih} (p_{21})_{hi} > 0$$

Secondly, since the irreducibility of the inter-impact between L_1 and L_2, for any fixed i, j∈ {1, 2, ..., n} there exists an integer q such that $t_{ij}^{(q)} > 0$, where $t_{ij}^{(q)}$ is the element of T^q. Consequently

$$t_{ij}^{(q+1)} = \sum_{k=1}^{n} t_{ik}^{(q)} t_{kj} \geq t_{ij}^{(q)} t_{jj} > 0$$

Therefore, $t_{ij}^{(u)} > 0$ for any integer u≥q. It means that there exists an integer r such that $T^r > 0$, which yields T as primitive (Gantmacher, 1977, Vol.II, p.80, Theorem 8). Similarly, $P_{21}P_{12}$ is primitive. The primitivity of $P_{12}P_{21}$ and $P_{21}P_{12}$ means that U^2 is regular or the cyclicity of U is 2. Thus, any column of the matrix

$$\tilde{U} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} U^k = \frac{1}{2}(E+U)(U^2)^{\infty} = \frac{1}{2} \begin{pmatrix} (P_{12}P_{21})^{\infty} & P_{12}(P_{21}P_{12})^{\infty} \\ P_{21}(P_{12}P_{21})^{\infty} & (P_{21}P_{12})^{\infty} \end{pmatrix} \tag{4}$$

is identical to any other column (Gantmacher, 1977, Vol.II, p.98). Let a be any column of $(P_{12}P_{21})^{\infty}$ and b be any column of $(P_{21}P_{12})^{\infty}$, respectively. They are both positive and uniquely satisfy Eq. (2) or Eq. (1) (Gantmacher, 1977, Vol.II, p.93 and p.98). This completes the proof.

Since Property A is true for a Bayes loop, we can also obtain the following corollary from Theorem 1.

Corollary 1: *If the inter-impact between L_1 and L_2 is an irreducible Bayes loop, then there exist two unique positive vectors a and b that satisfy Eq. (1).*

3. The Equivalent Conditions Between Eq. (1) and Definition 1

It is easy to see that the solution of Definition 1 is a solution of Eq. (1), but the solution of Eq. (1) is not always a solution of Definition 1 for arbitrary P_{12} and P_{21} with the conditions of Theorem 1. When would the solution of Eq. (1) be a solution of Definition 1? Or, given P_{21} and P_{12}, how can we find that the inter-impact between L_1 and L_2 is a Bayes loop? Theorem 2 answers this question. Also, a simple method of finding the solution of Eq. (1) will be introduced. First we introduce two lemmas for Theorem 2.

Lemma 1: *When the inter-impact between L_1 and L_2 is irreducible with Property A, the solution of Eq. (1) is a solution of Definition 1 if and only if, for coefficient matrices P_{12} and P_{21}, there exist positive constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$ such that*

$$\alpha_k (p_{12})_{ij} (p_{21})_{jk} = \alpha_i (p_{12})_{kj} (p_{21})_{ji} \quad \text{for } i, k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m \tag{5}$$

and

$$\beta_k (p_{21})_{ij} (p_{12})_{jk} = \beta_i (p_{21})_{kj} (p_{12})_{ji} \quad \text{for } i, k = 1, 2, \ldots m;\ j = 1, 2, \ldots, n \tag{6}$$

Proof (Necessity): If the solution of Eq. (1) is a solution of Definition 1, then the positive vectors $a = (a_1, a_2, \ldots, a_n)^T$ and $b = (b_1, b_2, \ldots, b_m)^T$ satisfy

$$(p_{12})_{ij} b_j = (p_{21})_{ji} a_i \qquad \text{for } i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m \tag{7}$$

and

$$(p_{21})_{jk} a_k = (p_{12})_{kj} b_j \qquad \text{for } k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m \tag{8}$$

Multiplying both sides of Eq.(7) by the corresponding sides of Eq.(8), we have

$$a_k (p_{12})_{ij} (p_{21})_{jk} = a_i (p_{12})_{kj} (p_{21})_{ji} \qquad \text{for } i, k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m \tag{9}$$

For the same reason, we also have

$$b_k (p_{21})_{ij} (p_{12})_{jk} = b_i (p_{21})_{kj} (p_{12})_{ji} \qquad \text{for } i, k = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{10}$$

Thus, the necessity follows.

(Sufficiency): Assume that there exist positive constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$ such that Eqs. (5) and (6) are true. Summing up Eq.(5) with respect to k, we obtain

$$(p_{12})_{ij} \sum_{k=1}^{n} (p_{21})_{jk} \alpha_k = (p_{21})_{ji} \alpha_i \qquad \text{for } i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m \tag{11}$$

Summing up Eq.(11) with respect to j, we obtain

$$\sum_{k=1}^{n} \Big(\sum_{j=1}^{m} (p_{12})_{ij} (p_{21})_{jk} \Big) \alpha_k = \alpha_i \qquad \text{for } i = 1, 2, \ldots, n \tag{12}$$

or

$$P_{12} P_{21} \alpha = \alpha \tag{13}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$. Similarly, we can obtain

$$P_{21}P_{12}\beta = \beta \tag{14}$$

where $\beta = (\beta_1, \beta_2, ..., \beta_m)^T$. From Eq. (13), it can be seen that

$$P_{21}P_{12}P_{21}\alpha = P_{21}\alpha \tag{15}$$

This means that $P_{21}\alpha$ is also the solution of Eq. (14). From the uniqueness of solution of Eq. (14), we obtain

$$\beta = P_{21}\alpha$$

or

$$\beta_j = \sum_{k=1}^{n} (p_{21})_{jk}\alpha_k \qquad \text{for } j = 1, 2, ..., m \tag{16}$$

Similarly, we can obtain

$$\alpha = P_{12}\beta$$

Thus, α and β constitute the solution of Eq. (1). Combining Eqs. (11) and (16), we obtain

$$(p_{12})_{ij}\beta_j = (p_{21})_{ji}\alpha_i \qquad \text{for } i = 1, 2, ..., n;\ j = 1, 2, ..., m \tag{17}$$

This means α and β also satisfy Definition 1. Therefore, the sufficiency follows.

It is not easy to check the conditions of Lemma 1. The following lemma will serve this purpose.

Lemma 2: *For non-negative matrices P_{12} and P_{21}, there exist positive constants $\alpha_1, \alpha_2, ..., \alpha_n$ and $\beta_1, \beta_2, ..., \beta_m$ such that*

$$\alpha_k (p_{12})_{ij} (p_{21})_{jk} = \alpha_i (p_{12})_{kj} (p_{21})_{ji} \qquad \text{for } i, k = 1, 2, ..., n;\ j = 1, 2, ..., m$$

and

$$\beta_k (p_{21})_{ij} (p_{12})_{jk} = \beta_i (p_{21})_{kj} (p_{12})_{ji} \qquad \text{for } i, k = 1, 2, ...m;\ j = 1, 2, ..., n$$

if and only if the ratios

$$\frac{(p_{12})_{ij}(p_{21})_{jk}}{(p_{12})_{kj}(p_{21})_{ji}} \qquad \text{for } (p_{12})_{kj}(p_{21})_{ji} \neq 0;\ i, k = 1, 2, ..., n;\ j = 1, 2, ..., m \tag{18}$$

and

$$\frac{(p_{21})_{ij}(p_{12})_{jk}}{(p_{21})_{kj}(p_{12})_{ji}} \qquad \text{for } (p_{21})_{kj}(p_{12})_{ji} \neq 0;\ i, k = 1, 2, ..., m;\ j = 1, 2, ..., n \tag{19}$$

do not depend on j.

Proof: The necessity is obvious. Now we prove the sufficiency. Suppose

$$\omega_{ik} = \frac{(p_{12})_{ij}(p_{21})_{jk}}{(p_{12})_{kj}(p_{21})_{ji}} \qquad \text{for } (p_{12})_{kj}(p_{21})_{ji} \neq 0;\ i, k = 1, 2, ..., n;\ j = 1, 2, ..., m \tag{20}$$

Since the ratios

$$\frac{(p_{12})_{ij}(p_{21})_{jk}}{(p_{12})_{kj}(p_{21})_{ji}} \qquad \text{for } (p_{12})_{kj}(p_{21})_{ji} \neq 0;\ i, k = 1, 2, ..., n;\ j = 1, 2, ..., m$$

do not depend on j, we see that

$$\omega_{ik} = \omega_{it}\,\omega_{tk} \qquad \text{for } i, k, t = 1, 2, ..., n$$

Therefor, the matrix $A = (\omega_{ik})$ is consistent and there exist positive constants $\alpha_1, \alpha_2, ..., \alpha_n$ such that (Saaty, 1990)

$$\omega_{ik} = \frac{\alpha_i}{\alpha_k} \qquad i, k = 1, 2, ..., n \tag{21}$$

From Eqs. (20) and (21) we obtain

$$\frac{\alpha_i}{\alpha_k} = \frac{(p_{12})_{ij}(p_{21})_{jk}}{(p_{12})_{kj}(p_{21})_{ji}} \qquad \text{for } (p_{12})_{kj}(p_{21})_{ji} \neq 0;\ i, k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m$$

or

$$\alpha_k(p_{12})_{ij}(p_{21})_{jk} = \alpha_i(p_{12})_{kj}(p_{21})_{ji} \qquad \text{for } i, k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m$$

Similarly, we can show that there exist positive constants $\beta_1, \beta_2, \ldots, \beta_m$ such that

$$\beta_k(p_{21})_{ij}(p_{12})_{jk} = \beta_i(p_{21})_{kj}(p_{12})_{ji} \qquad \text{for } i, k = 1, 2, \ldots m;\ j = 1, 2, \ldots, n$$

Hence the sufficiency follows.

Combining Lemmas 1 and 2, we have the following theorem.

Theorem 2: *When the inter-impact between L_1 and L_2 is irreducible with Property A, the solution of Eq. (1) is a solution of Definition 1 if and only if, for coefficient matrices P_{12} and P_{21}, the ratios*

$$\frac{(p_{12})_{ij}(p_{21})_{jk}}{(p_{12})_{kj}(p_{21})_{ji}} \qquad \text{for } (p_{12})_{kj}(p_{21})_{ji} \neq 0;\ i, k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m$$

and

$$\frac{(p_{21})_{ij}(p_{12})_{jk}}{(p_{21})_{kj}(p_{12})_{ji}} \qquad \text{for } (p_{21})_{kj}(p_{12})_{ji} \neq 0;\ i, k = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$$

do not depend on j.

Corollary 2: *The inter-impact between L1 and L2 is a Bayes loop if and only if, for matrices P_{12} and P_{21}, the ratios*

$$\frac{(p_{12})_{ij}(p_{21})_{jk}}{(p_{12})_{kj}(p_{21})_{ji}} \qquad \text{for } (p_{12})_{kj}(p_{21})_{ji} \neq 0;\ i, k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m$$

and

$$\frac{(p_{21})_{ij}(p_{12})_{jk}}{(p_{21})_{kj}(p_{12})_{ji}} \qquad \text{for } (p_{21})_{kj}(p_{12})_{ji} \neq 0;\ i, k = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$$

do not depend on j.

Assume that $L_1 = \{\theta_1, \theta_2, \ldots, \theta_n\}$ is the set of states of nature and let $L_2 = \{x_1, x_2, \ldots, x_m\}$ be the sample space from which observations are drawn at random. Let $P_{21} = ((p_{21})_{ij}) = (p(x_i|\theta_j))$ be the m×n column stochastic matrix of likelihoods, let $P_{12} = ((p_{12})_{ij}) = (p(\theta_i|x_j))$ be the n×m column stochastic matrix of posterior probabilities, $P_1 = ((p_1)_i) = (p(\theta_i))$ be the n×1 vector of prior probabilities, and $P_2 = ((p_2)_i) = (p(x_i))$ be the m×1 vector of the marginal probabilities of x_i. We have

$$p(\theta_i|x_j)p(x_j) = p(x_j|\theta_i)p(\theta_i) \qquad \text{for } i=1, 2, \ldots, n;\ j = 1, 2, \ldots, m$$

that is, Definition 1 is satisfied. By Theorem 2, we have the following corollary.

Corollary 3: *The ratios*

$$\frac{p(\theta_i|x_j)p(x_j|\theta_k)}{p(\theta_k|x_j)p(x_j|\theta_i)} \ \textit{and}\ \frac{p(x_i|\theta_j)p(\theta_j|x_k)}{p(x_k|\theta_j)p(\theta_j|x_i)}$$

do not depend on j.

Using Theorem 2, we find a simple method of finding a and b that satisfy Eq. (1) when the conditions of Theorem 2 are satisfied. In fact, in this situation there exist positive constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$ such that

$$\frac{\alpha_i}{\alpha_k} = \frac{(p_{12})_{ij}(p_{21})_{jk}}{(p_{12})_{kj}(p_{21})_{ji}} \quad \text{for } (p_{12})_{kj}(p_{21})_{ji} \neq 0;\ i, k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m \tag{22}$$

and

$$\frac{\beta_i}{\beta_k} = \frac{(p_{21})_{ij}(p_{12})_{jk}}{(p_{21})_{kj}(p_{12})_{ji}} \quad \text{for } (p_{21})_{kj}(p_{12})_{ji} \neq 0;\ i, k = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{23}$$

From the ratios of the right side of Eq. (22) and those of the right side of Eq. (23) we can obtain (except a constant factor) $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$, respectively. Normalizing them we can obtain a and b. Here is an example.

Example 3: Suppose, in Eq. (1),

$$P_{12} = \begin{pmatrix} 0.4167 & 0.5769 & 0.7143 & 0.2273 \\ 0.2500 & 0.1154 & 0.2143 & 0.6818 \\ 0.3333 & 0.3077 & 0.0714 & 0.0909 \end{pmatrix} \text{ and } P_{21} = \begin{pmatrix} 0.2 & 0.2 & 0.4 \\ 0.3 & 0.1 & 0.4 \\ 0.4 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.1 \end{pmatrix}$$

It is easy to verify that the conditions of Theorem 2 are satisfied. Then, by normalizing the ratios

$$\frac{a_2}{a_1} = \frac{(p_{12})_{2j}(p_{21})_{j1}}{(p_{12})_{1j}(p_{21})_{j2}} = 0.6, \quad \frac{a_3}{a_1} = \frac{(p_{12})_{3j}(p_{21})_{j1}}{(p_{12})_{1j}(p_{21})_{j3}} = 0.4 \text{ and } \frac{a_1}{a_1} = 1$$

we obtain $a_1 = 0.5$, $a_2 = 0.3$ and $a_3 = 0.2$. Also, by normalizing the ratios

$$\frac{b_2}{b_1} = \frac{(p_{21})_{2j}(p_{12})_{j1}}{(p_{21})_{1j}(p_{12})_{j2}} = 1.0833, \quad \frac{b_3}{b_1} = \frac{(p_{21})_{3j}(p_{12})_{j1}}{(p_{21})_{1j}(p_{12})_{j3}} = 1.1667,$$

$$\frac{b_4}{b_1} = \frac{(p_{21})_{4j}(p_{12})_{j1}}{(p_{21})_{1j}(p_{12})_{j4}} = 0.9167 \text{ and } \frac{b_1}{b_1} = 1$$

we obtain $b_1 = 0.24$, $b_2 = 0.26$, $b_3 = 0.28$ and $b_4 = 0.22$. Thus the vectors $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3, b_4)^T$ form the solution of Eq. (1).

4. Conclusion

In this paper, some problems about the solution of a superhierarchy with a Bayes loop are discussed and a simple method of finding the solution is introduced. Theorem 2 shows the relationship between the likelihoods ($(p_{21})_{ij}$) and the posterior probabilities ($(p_{12})_{ij}$). It is a supplement to Bayes Theorem. The results obtained here can be extended to some more general situations (a superhierarchy with several Bayes loops, for example), which can be the topics for further research.

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