Inner and Outer Dependence in the Analytic Hierarchy Process : The Supermatrix and the Superhierarchy

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April 1991

1 - Introduction

How does one structure a decision problem with dependencies, derive priorities and make choices among interdependent alternatives?

Dependence is a primitive concept that has two meanings.

1) Outer dependence or the dependence of an alternative <u>on an attribute</u> possessed by many or all of the alternatives, is the degree or intensity to which that attribute is present in the alternative (one car has a lot of style and another little or no style) either a) measured relative to other alternatives, or b) measured alone on a standard. In the end both results can be expressed in relative terms for the alternatives considered; the first is already relative and the second becomes relative after normalization. This is why we shall not distinguish between these two possibly differently derived results in our analysis of dependence. Industries use different amounts of electricity so that some are heavy users and others are light users. It is not possible to distinguish between the two clusters unless they are given together in one group.

Another form of outer dependence occurs in the opposite way. Which attribute of several is in this alternative more? We note that attributes derive from their objects and become mental abstractions only after the objects are experienced. Again, an attribute is either measured in relative terms by comparing it with other attributes on higher order criteria, or (in rare cases) absolutely by measuring it on a scale where there is one for that attribute and deciding whether it is high or low and then interpreting the magnitude of the result as compared with that of another attribute. It is important to note that people do this kind of thinking. For example, this apple is more red than it is tasty.

2) Inner dependence, or the dependence of an alternative on another alternative, is the influence, contribution or impact of the second alternative on the first with respect to (conditional on) an attribute they have in common. For example, all industries depend on the electric industry for their lighting. The electric industry itself depends on the oil, coal and nuclear industries for raw material which in turn use electricity to produce their goods. An alternative may depend on another with respect to several attributes. A child depends on its family for physical and social care. Conversely, the family depends on the child for its self realization and for physical help.

To study dependence in decision problems we need to know both the alternatives and the attributes and what depends on what, (we call this functional dependence) and also the precise way in which these elements are located and their interconnections made (we call this structural dependence.) Let us make some general observations about what we are looking for to cope with dependence. A mathematical model is limited by its structure: it can be linear or nonlinear, normative or descriptive, have or not have dependencies among the variables; and by what operations one needs to carry out to derive useful results. A crucial property of a model is its adaptability to represent complex problems and its faithfulness to reflect the degree and variety of dependencies among the variables considered. Our understanding and control of the world depend on our ability to develop techniques of modelling to capture awareness of a problem faithfully. In addition, whatever the model, we need to consider the quality of the answers we get from it.

Figure 1, taken from the New York Times, Sunday, December 12, 1976 illustrates how naturally dependence occurs in the real world. It is a mess. We need to set priorities in a situation like this to understand the net cause and effect and where to control it. The theory we develop here is designed for that purpose.

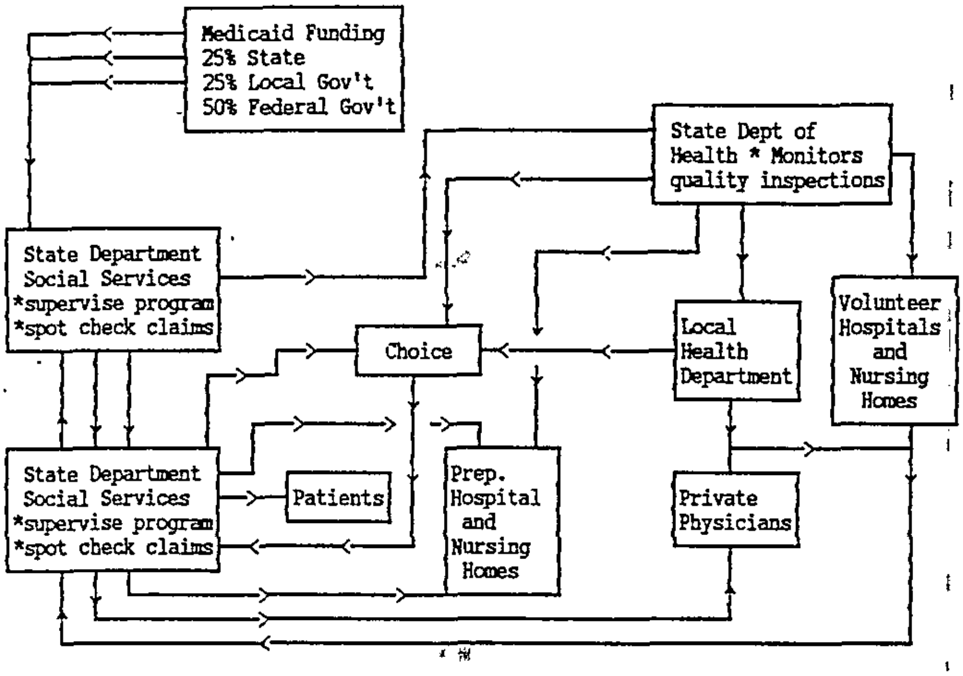

Figure 1

2 - The Structure of Decision Problems

A hierarchy is a linear structure used in decision theory to represent the simplest type of dependence of one level or component of a system on another in a sequential manner. Usually the top level of a hierarchy is a single element, the overall goal, from which influence emanates to the next level below. The remaining levels each have several elements to which influence flows from the level above.

To determine these influences we must perform pairwise comparisons on the elements with respect to the goal. The human brain does not measure a steady state, it only responds to change. If one puts one hand in hot water and the other in cold water and then puts both hands in tepid water, it will feel cold to the first and hot to the second. The skin does not measure temperature but only changes in temperature. To test changes, the brain constantly compares. Thus paired comparisons is fundamental in setting priorities for a system with dependencies and interactions.

A hierarchy is said to be complete or incomplete depending on whether or not all the elements in a lower level depend on all the elements in the adjacent level above. A hierarchy is a convenient way to decompose a complex problem in search of cause-effect explanations in steps which form a linear chain. Sometimes outer dependence can take place between the bottom level of a hierarchy and its top level, when the top level is not a single goal, but several objectives whose priorities depend on the lowest level, the alternatives of the decision. Such a hierarchy is known as a holarchy. For example, in a decision on terrorism, the objectives, in the top level, dictated the most effective method to prevent or respond to terrorist acts in the bottom level. Conversely, each response highlighted the most likely objective to fulfill and the totality of responses indicated the overall objective to be fulfilled.

A more general way to structure a problem involving dependence which allows for feedback between components is a network system of which a hierarchy is a special case. A network involves, for example, dependence of a component or cluster A of elements, (the counterpart of a level in a hierarchy) on another collection of elements B and also the dependence of B on A. Dependence between components may follow a cycle that returns to its starting component. There may be several cycles with overlap. Clearly, the components of a network cannot be labeled higher or lower, as can the levels of a hierarchy.

For a network system, interaction between two components, just as in the case of the levels of a hierarchy, may be characterized as complete or incomplete. To represent a network, we may simply use an arc (a directed edge) from one component to another to show the order of flow of influence between components. Two interactive components are indicated by two arcs going in opposite directions. Sometimes it may happen that the elements of a component are dependent on other elements in the same component. This is represented by a loop, an arrow from the component back to it, indicating inner dependence of elements within that component with respect to an attribute in the adjacent components, i.e. components with an arrow directed to that component. Figure 2 below depicts the structural difference between the two frameworks, hierarchies and feedback systems.

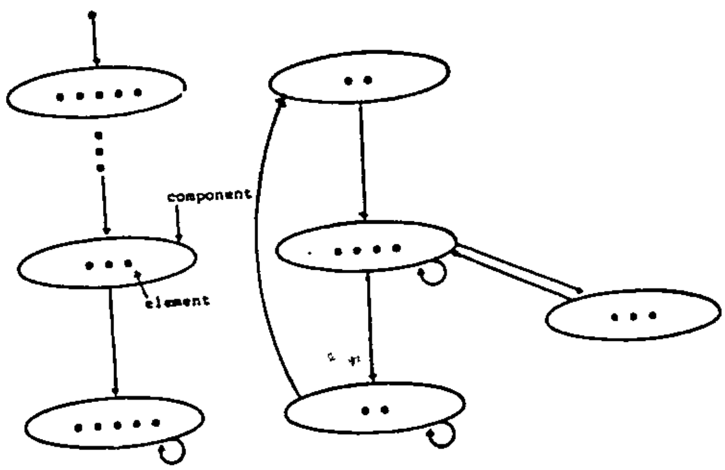

Figure 2

To represent the interaction of elements within two adjacent components, a 0,1 matrix is used for each such pair. Therefore for each pair of components we have either one or two 0,1 matrices. The totality of these matrices is a detailed representation of the flow of influence.

Our problem here is to show how to derive priorities in such systems. We shall be mainly interested in systems in which all the elements in a component are taken together with respect to every element of another component as in a complete hierarchy. It will turn out that the case of an incomplete hierarchy is a special case of this, with zero adjoined for the missing parts.

3 - So Far a Single Form of Dependence

In the literature one rarely finds a distinction drawn between inner and outer dependence. It is simply treated as dependence.

W. Leontief's practical notion of input-output analysis among industries in the field of economics, and the dependence among events in probability theory. There are even more complex dependencies of stochastic processes found in Markov and Semi-Markov processes. All these are specialized attempts for dealing with dependence.

The notion of dependence also occurs in utility theory where the idea of a lottery plays an important role. Here, Y is defined to be utility independent of Z when conditional preferences for lotteries on Y given $z \in Z$ do not depend on the particular level of z. One theoretical objective when using utility theory is to derive a utility function, but the existence of such a function is conditioned by the independence of criteria from alternatives. In addition one ordinarily assumes that the criteria are independent among themselves and similarly for the alternatives.

4 - Priorities in Systems - Outer Dependence

A system is decomposable if its elements can be aggregated into components whose interactions are represented by the arcs of a directed network. In this case we derive the priorities between the elements of adjacent components as in a hierarchy.

Because the components of a system, and hence also the elements in these components, can interact along more than a single path, the priorities of influence of a component of the system on another component may be measured over all the paths and cycles which connect them. Note that if elements in one component depend on the elements of another then we can also speak of dependence between the two components. However, two components may depend on each other due to the synergy of their elements without the elements themselves being directly dependent. For example, two industries may depend on each other's output without their machines depending on each other directly.

a. Outer Dependence of Two Components

The simplest example of outer dependence is between two components one consisting of the **m** alternatives which depend on the other of **n** criteria and vice versa, the dependence of the criteria on the alternatives. As usual, vectors of priorities are derived for the alternatives in terms of each of the criteria. These vectors can be arranged in an **m by n** matrix **A** which represents the influence of the criteria on the alternatives. Similarly vectors of priorities for the criteria with respect to the alternatives are derived and arranged as columns of an **n by m** matrix **B**. The two matrices are incorporated as blocks of an **n+m** by **n+m** supermatrix which represents the impact of the elements of the two components C_1 and C_2 of criteria and alternatives respectively on the elements themselves of the two components C_1 and C_2. There are four block matrices in the supermatrix. The diagonal blocks represent the impact of the elements of each component on themselves and has zero values. The off diagonal blocks **A** and **B** represent the interaction (C_2,C_1) and (C_1,C_2) respectively. The supermatrix has the form

$$\begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \end{array}$$

This matrix is irreducible because the two components are enduring (or recurrent) [10]. It is column stochastic which means that its columns sum to unity. Here the columns of **A** and **B** are normalized eigenvectors and hence sum to unity. By raising the supermatrix to powers one captures the dominance among the elements over all the paths and cycles in the network in a limiting matrix. In this matrix, the columns in the **A** block position become identical and thus any one of them yields the limiting priorities of the alternatives in component C_2. Similarly, the columns in the B block position are identical and any one of them yields the limiting priorities of the criteria in component C_1.

Example

The following illustration has to do with the management of a water reservoir. Here we are faced with the decision to choose one of the possibilities of maintaining the water level in a dam at: Low (L), Medium (M) or High (H) depending on the relative importance of Flood Control (F), Recreation (R) and the generation of Hydroelectric Power (E) respectively for the three levels. The first set of three matrices gives the prioritization of the alternatives with respect to the criteria and the second set, those of the criteria in terms of the alternatives.

Which level is best for flood control?

Flood Control	Low	Med	Hi	Eigen-vector
Low	1	5	7	.722
Medium	1/5	1	4	.205
High	1/7	1/4	1	.073

Consistency Ratio .107

Which level is best for recreation?

Recreation	Low	Med	Hi	Eigen-vector
Low	1	1/7	1/5	.072
Medium	7	1	3	.649
High	5	1/3	1	.279

Consistency Ratio .056

Which level is best for power generation?

Hydro-electric Power	Low	Med	Hi	Eigen-vector
Low	1	1/5	1/9	.058
Medium	5	1	1/5	.207
High	9	5	1	.735

Consistency Ratio .101

At Low Level which attribute is satisfied best?

Low Level Dam	F	R	E	Eigen-vector
Flood control	1	3	5	.637
Recreation	1/3	1	3	.258
Hydroelectric power	1/5	1/3	1	.105

Consistency Ratio .033

At Intermediate Level which attribute is satisfied best?

Intermediate Dam	F	R	E	Eigen-vector
F	1	1/3	1	.200
R	3	1	3	.600
E	1	1/3	1	.200

Consistency Ratio .000

At High Level which attribute is satisfied best?

High Level Dam	F	E	R	Eigen-vector
Flood control	1	1/5	1/9	.060
Recreation	5	1	1/4	.231
Hydroelectric power	9	4	1/6	.709

Consistency Ratio .061

As in the previous example, the supermatrix is given by:

	F	R	E	L	M	H
F	0	0	0	.637	.200	.060
R	0	0	0	.258	.600	.231
E	0	0	0	.105	.200	.709
L	.722	.072	.058	0	0	0
M	.205	.649	.207	0	0	0
H	.073	.279	.735	0	0	0

and the limiting matrix of powers of this supermatrix by:

	F	R	E	L	M	H
F	0	0	0	.241	.241	.241
R	0	0	0	.374	.374	.374
E	0	0	0	.385	.385	.385
L	.223	.223	.223	0	0	0
M	.372	.372	.372	0	0	0
H	.405	.405	.405	0	0	0

This matrix shows a preference for a high dam with priority .405 for hydroelectric power generation with priority .385.

b. The General Case of Outer Dependence

Every system has a purpose that arises out of how that system is set up. A television set has the purpose of producing good pictures and good sound. Behind this purpose there are higher order purposes such as economy and efficiency of operation, reliability and so on. The performance of each part is evaluated in terms of how these factors are incorporated in its design so that the system has some redundancy. For example, its components should be located in some optimal way and it should have little or no undesirable side effects such as radiation, heat and interference with other systems in the house. These higher order purposes are usually incorporated by choosing the appropriate material and by manufacturing the parts in a certain way then locating them in the network in the best way to fulfill their function to satisfy all the higher order purposes which we might call supercriteria. It will be seen below that in setting priorities in systems with feedback, these higher order criteria and their priorities play an important role.

Now let us look at the process of generating priorities for a general system with outer dependence. In simple terms, there are three categories of components in this system. A source component (for example of criteria) with no arc entering the component (there may be several such components); intermediate components (of subcriteria) with arcs entering and others exiting (flow in such components may repeat if they belong to cycles) ; and an end or sink component (of alternatives) with arcs only entering and none exiting. But there are refinements of these ideas best understood by studying the literature (see for example [10]). A supermatrix for the several components is defined as before. The block matrices of eigenvectors as in **A** and **B** discussed earlier are determined and inserted in the appropriate position in the supermatrix to indicate interaction between the elements of their respective components. In positions where there is no interaction, a zero block matrix is introduced. The diagonal block matrices correspond to the interaction of a component with itself. These are zero except if the component is a sink with arrows entering but none exiting. In that case the block must correspond to the identity matrix. Thus for example, in the supermatrix corresponding to a hierarchy, the nonzero blocks occupy the subdiagonal positions with zeros everywhere else except for an identity matrix in the last row and column block position which corresponds to the elements of the last level in the sink component.

To obtain limiting priorities we need to ensure that the supermatrix is column stochastic. In this case, because there may be several nonzero blocks under a component, the sum of the entries in each column can be an integer equal to the number of nonzero blocks since the column in each block is a normalized eigenvector. To make the supermatrix stochastic, we must set priorities on the row components themselves as they influence each column component. To do this we need to introduce a three level control (or design) hierarchy in which the third level elements are these components of the supermatrix. In the second level of the control hierarchy are supercriteria mentioned earlier. The top level supergoal of this hierarchy is the satisfactory functioning of the system. The supercriteria to ensure this satisfactory functioning may be economy, reliability, replaceability, usability; or in another case economic, environmental, and social factors, and so on. We first obtain their priorities in the usual hierarchical way.

Some or all of the components in the third level influence each component (inner dependence) as would be known from the arcs of the network. For each supercriterion of the control hierarchy, inner dependence (see part 6 on inner dependence) eigenvectors are derived for the components of the supermatrix by considering how strongly the components of the supermatrix contribute to each component with respect to that criterion. Each of these vectors is used to weight the block matrix of the supermatrix in the column corresponding to the dependent component. As a result of this weighting the supermatrix becomes stochastic. Depending on whether the supermatrix is reducible or not, there is a theory [10] for the existence of a limiting matrix and corresponding priorities.

Naturally the influence needs to be considered for each of the criteria in the superhierarchy. Thus for each supercriterion, we obtain different weighting for the components in the supermatrix. The limiting priorities derived from each supermatrix are weighted by the importance of the supercriterion with respect to which the priorities of the components are determined and the results are added. This is accomplished by multiplying every element in the limiting supermatrix by the single priority number of the criterion and the supermatrices are added to obtain a single composite supermatrix. This yields the overall outer dependence priorities in the network.

5 - How to Use Lotus 1-2-3 for Outer Dependence

In this section we describe the steps to follow in using Lotus 1-2-3 to set up the supermatrix and calculate the limiting priorities.

1. Enter Lotus 1-2-3.

2. Fill in the matrix.

3. Press / (and a menu will appear).

4. Sèlect DATA (Press D),
then MATRIX (Press M),
then MULTIPLY (Press M).

5. The computer will ask you to enter the First Range to Multiply.

- Put the cursor in the first cell of the matrix using the arrow keys.
- Press .(period) to fix the first part of the range.
- Using the arrow keys to block the entire matrix.
- When you are finished Press Enter.

You have now blocked the first matrix.

6. The computer will now ask you to enter the Second Range to Multiply.

- Do as in Step 5 by reblocking the same matrix. You have now blocked the second matrix to multiply the first matrix by.

7. The computer will now ask to select the Output Range.

- Position the cursor below the original matrix leaving at least one blank row and Press Enter.

The product of the two matrices now appears.

8. Repeat Steps 3 through 6, except when you have to select the second range to multiply, first press Escape, then blockout the product matrix.

9. From then on all one does is Press /,D,M,M, Enter, Enter, and Enter until the desired accuracy is reached when all the columns of each block of the product matrix are the same.

Note : This provides a single increment of power for each cycle. To speed the process, multiply the product matrix by itself repeatedly. In the end, stop if you want even powers or multiply the result by the original matrix for an odd power result.

6 - Priorities in Systems - Inner Dependence Loops

There are two types of inner dependence. One is determined solely by the internal relations among the alternatives with respect to each criterion. We call this inner-inner dependence. For example in family-life, housekeeping responsibilities and other duties are shared, creating dependencies undertaken by the different members with various degrees of leadership and performance to satisfy the needs of the family. The fact that the father may be the president of the United States would not matter very much to his standing in his family. He still must fulfill his family objectives.

In the other type of inner dependence, priorities are weighted by the outer dependence priorities of the members. We call this inner-outer dependence. The President has outside duties and some of his internal responsibilities are taken over by someone else in order that he may have sufficient energy to pursue his outside duties. Thus for each member, his or her contributions to each other is weighted by the relative outer priority of the family members obtained from a hierarchy or a supermatrix if necessary, treating each as if he or she is independent of the others. The goal of that hierarchy is making the most contributions to society. In this manner one obtains the independence outer priorities for each member of the family.

For the inner dependence priorities, one takes each criterion, and for each member one constructs a paired comparison matrix to represent how much each member, including the dependent member in question, contributes to that member. For example in a family that member may not cook for himself but regularly washes the dishes for everybody. The process is repeated for all members and for every criterion.

a. Inner-Outer Dependence

We begin with calculations of priorities with inner-outer dependence because its mathematics is easier to follow. In general, the dependence of alternatives in a component on other alternatives in that component can be designated by a 0,1 matrix with zero indicating no dependence. Next for each alternative, a pairwise comparison matrix is used to compare in pairs the relative contribution of all the alternatives on which the given alternative depends (indicated by one in the 0,1 matrix), with respect to a criterion in the next higher level. This matrix gives rise to a vector of dependence priorities augmented by zeros in those positions which correspond to alternatives from which the given alternative is independent. The vectors thus generated, one for each alternative, will serve as the columns of a matrix of inner dependence vectors of the alternatives with respect to the criterion. To compute the interdependence priorities of the alternatives, this matrix is then multiplied on the right by the column vector of outer dependence priorities of the alternatives with respect to that criterion (obtained in the usual comparison of the alternatives as if they are independent.) The resulting vector is an interdependence vector of priorities of the alternatives with respect to the criterion. The process is then repeated to derive interdependence vectors for the alternatives with respect to each of the criteria. The resulting matrix of interdependence column vectors is then multiplied on the right by the vector of priorities of the criteria. This yields the overall interdependence vector of the alternatives.

b. Inner-Inner Dependence

If instead of using the outer dependence priority vector, we were to seek a purely inner dependence priority for the alternatives, we could weight each dependence vector by the (yet unknown) inner dependence weight of the dependent alternative. To determine such inner dependence of the alternatives one must solve an eigenvalue problem. The reason is that for each alternative, its priority is equal to a sum of weighted priorities. These are the product of its relative contribution to the first alternative times the priority of the first alternative, plus the product of its relative contribution to the second alternative times the priority of the second alternative and so on. This is an eigenvalue problem with maximum eigenvalue equal to one. A different eigenvector is derived for each criterion. The eigenvectors form the columns of a matrix which when weighted (multiplied on the right) by the priority vector of the criteria yields an overall inner dependence vector.

c. Hybrid Dependence

These two types of priorities can be combined (through elementwise multiplication) to produce a hybrid priority vector. We seldom bring inner interdependence to modify out dependence priorities. It is rare in our society that one knows whether an individual who is a prominent member in society is also prominent in family relations, is gentle or cruel, wise or foolish, has friends and is kind to animals. But with this kind of approach we can do more of it.

7 - Examples

Example 1 [1]

Consider the problem of choosing, in the next decade, between fuel alternatives for motor cars from a list of leading alternative fuels. This problem can be viewed in two ways. One is to do a cost benefits analysis of the competing alternatives and determine which is best after evaluating them in light of various criteria. This would tell us what the optimal choice might be. Another way to view the problem is to look at the various actors and forces that shape the outcome and see how they interact with each other to influence that outcome. This will predict what will be chosen rather than what is the best choice. With the many competing actors and forces interacting with each other, the problem is a good example for a system formulation. To save on detail the network of Figure 3 gives a sketch of the various elements of the problem. The supermatrix gives the relevant eigenvectors of the interacting components. What we need to explain is how to weight the components. For that purpose we have the control hierarchy of Figure 4. The three factors of level two of this hierarchy: Environmental, Economic and Social were assigned equal priorities.

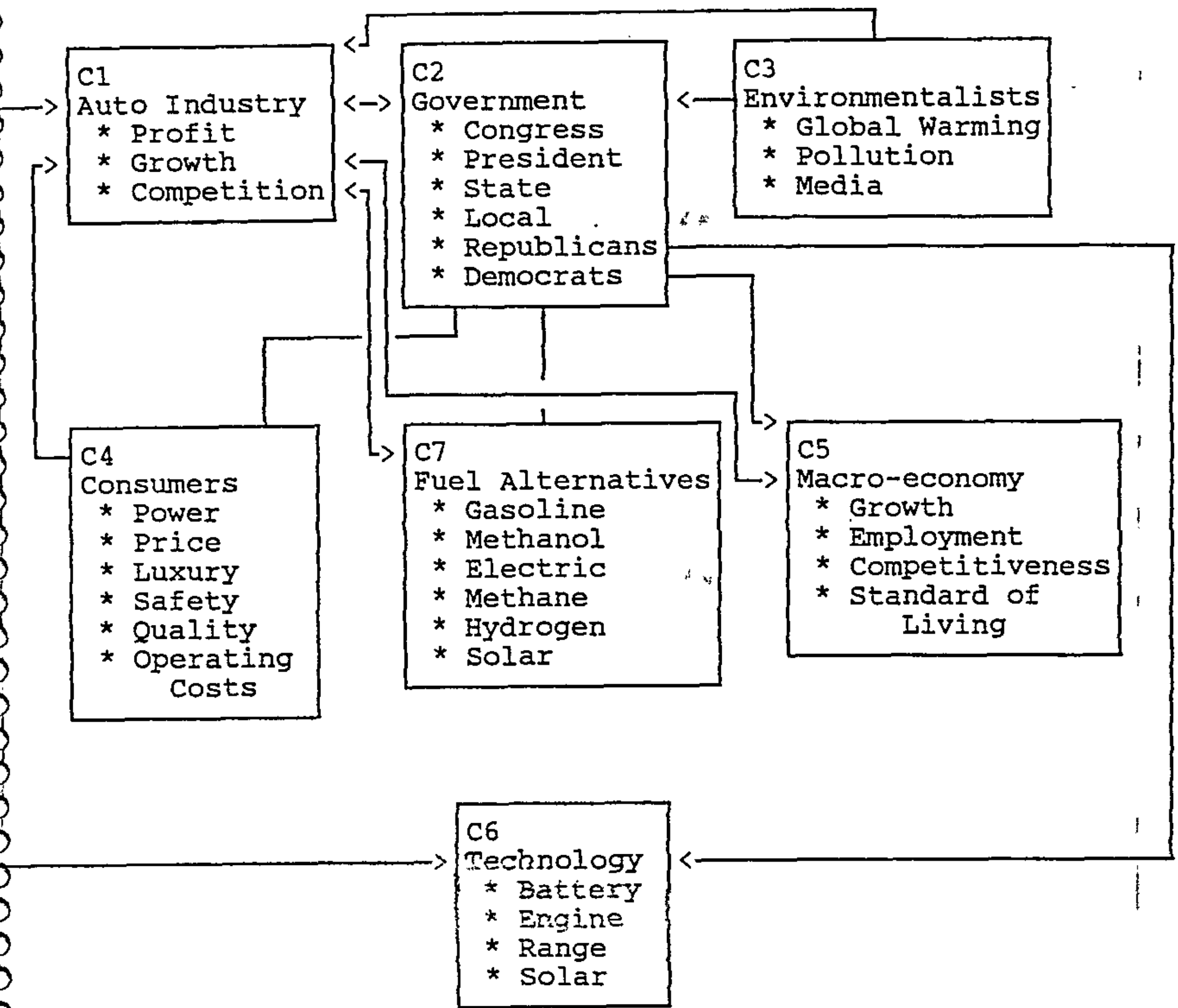

Figure 3 Network

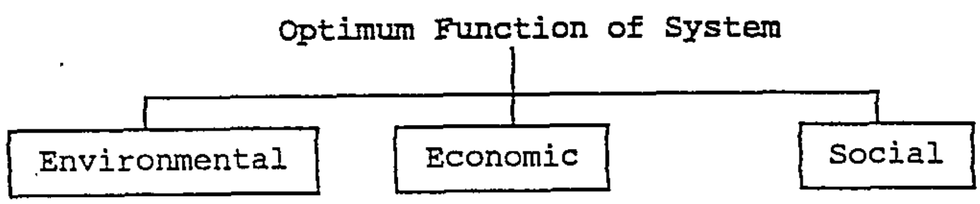

Figure 4 Control Hierarchy

The synthesized inner dependence priorities of the components with respect to the three criteria of the control hierarchy are shown in Table 1 below.

Alpha Weights Matrix

	AI	G	E	C	ME	T	FA
Automobile Industry	0.000	0.216	0.353	0.469	0.433	1.000	1.000
Government	0.416	0.000	0.647	0.531	0.567	0.000	0.000
Enviromentalists	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Consumers	0.000	0.362	0.000	0.000	0.000	0.000	0.000
Macro-Economy	0.230	0.218	0.000	0.000	0.000	0.000	0.000
Technology	0.205	0.118	0.000	0.000	0.000	0.000	0.000
Feul Alternatives	0.149	0.176	0.000	0.000	0.000	0.000	0.000

Table 1

These vectors of weights were used to weight the eigenvectors in the corresponding blocks of the supermatrix shown in Table 2. The resulting stochastic supermatrix was then raised to limiting powers. The priorities for the various fuels are shown below.

Fuel Alternative Results

Total Fuel Alternative Node Weight - 0.09423

Relative Fuel Alternative Weights	
Gasoline	0.13759
Methanol	0.20030
Electric	0.21533
Methane	0.21946
Hydrogen	0.10497
Solar	0.12235

Table 2

		P	G	C	C	P	S	L	R	D	GW	P	M	Po	Pr
	Profit	0.000	0.000	0.000	0.075	0.075	0.075	0.128	0.582	0.072	0.111	0.111	0.157	0.091	0.091
Auto Industry	Growth	0.000	0.000	0.000	0.696	0.896	0.896	0.595	0.309	0.649	0.111	0.111	0.248	0.091	0.091
	Competition	0.000	0.000	0.000	0.229	0.229	0.229	0.276	0.109	0.279	0.778	0.778	0.594	0.818	0.818
	Congress	0.150	0.198	0.360	0.000	0.000	0.000	0.000	0.000	0.000	0.334	0.235	0.381	0.310	0.441
	President	0.212	0.161	0.114	0.000	0.000	0.000	0.000	0.000	0.000	0.334	0.261	0.348	0.119	0.188
Government	State	0.130	0.108	0.099	0.000	0.000	0.000	0.000	0.000	0.000	0.154	0.304	0.157	0.310	0.129
	Local	0.079	0.065	0.099	0.000	0.000	0.000	0.000	0.000	0.000	0.042	0.043	0.097	0.069	0.063
	Republican	0.378	0.418	0.288	0.000	0.000	0.000	0.000	0.000	0.000	0.080	0.070	0.058	0.098	0.090
	Democrat	0.053	0.049	0.032	0.000	0.000	0.000	0.000	0.000	0.000	0.075	0.082	0.059	0.094	0.090
	Global Warming	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Envirmentalists	Pollution	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Media	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Power	0.000	0.000	0.000	0.047	0.047	0.047	0.047	0.047	0.047	0.000	0.000	0.000	0.000	0.000
	Price	0.000	0.000	0.000	0.212	0.212	0.212	0.212	0.212	0.212	0.000	0.000	0.000	0.000	0.000
Consumers	Luxury	0.000	0.000	0.000	0.047	0.047	0.047	0.047	0.047	0.047	0.000	0.000	0.000	0.000	0.000
	Safety	0.000	0.000	0.000	0.445	0.445	0.445	0.445	0.445	0.445	0.000	0.000	0.000	0.000	0.000
	Quality	0.000	0.000	0.000	0.173	0.173	0.173	0.173	0.173	0.173	0.000	0.000	0.000	0.000	0.000
	Operating Costs	0.000	0.000	0.000	0.075	0.075	0.075	0.075	0.075	0.075	0.000	0.000	0.000	0.000	0.000
	Growth	0.504	0.635	0.125	0.189	0.321	0.531	0.578	0.658	0.298	0.000	0.000	0.000	0.000	0.000
Macro-Economy	Employment	0.098	0.078	0.125	0.642	0.453	0.350	0.328	0.136	0.414	0.000	0.000	0.000	0.000	0.000
	Competitive	0.347	0.079	0.825	0.047	0.083	0.048	0.047	0.075	0.045	0.000	0.000	0.000	0.000	0.000
	Standard of Living	0.053	0.207	0.125	0.122	0.143	0.074	0.047	0.131	0.045	0.000	0.000	0.000	0.000	0.000
	Battery	0.346	0.250	0.098	0.313	0.313	0.308	0.268	0.313	0.250	0.000	0.000	0.000	0.000	0.000
Technology	Range	0.424	0.250	0.190	0.313	0.313	0.308	0.268	0.313	0.250	0.000	0.000	0.000	0.000	0.000
	Engine	0.181	0.250	0.643	0.313	0.313	0.308	0.268	0.313	0.250	0.000	0.000	0.000	0.000	0.000
	Solar	0.049	0.250	0.071	0.063	0.063	0.077	0.143	0.063	0.250	0.000	0.000	0.000	0.000	0.000
	Gasoline	0.404	0.167	0.187	0.044	0.049	0.038	0.047	0.049	0.036	0.000	0.000	0.000	0.000	0.000
	Methanol	0.174	0.187	0.167	0.282	0.254	0.180	0.084	0.281	0.180	0.000	0.000	0.000	0.000	0.000
Feul Alternatives	Electric	0.174	0.187	0.167	0.282	0.254	0.352	0.447	0.158	0.352	0.000	0.000	0.000	0.000	0.000
	Methane	0.174	0.187	0.167	0.262	0.254	0.248	0.091	0.370	0.248	0.000	0.000	0.000	0.000	0.000
	Hydrogen	0.041	0.187	0.167	0.087	0.078	0.074	0.143	0.053	0.074	0.000	0.000	0.000	0.000	0.000
	Solar	0.033	0.187	0.187	0.104	0.110	0.109	0.209	0.087	0.109	0.000	0.000	0.000	0.000	0.000

		L	S	Q	CC	G	E	C	SL	B	R	E	S	G	M	E	M	H	S
	Profit	0.091	0.091	0.091	0.091	0.218	0.228	0.109	0.167	0.333	0.400	0.200	0.200	0.333	0.333	0.333	0.333	0.333	0.333
Auto Industry	Growth	0.091	0.091	0.091	0.091	0.630	0.674	0.183	0.167	0.333	0.200	0.117	0.117	0.333	0.333	0.333	0.333	0.333	0.333
	Competition	0.818	0.818	0.818	0.818	0.151	0.101	0.729	0.167	0.333	0.400	0.683	0.683	0.333	0.333	0.333	0.333	0.333	0.333
	Congress	0.369	0.441	0.369	0.180	0.217	0.275	0.200	0.275	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	President	0.215	0.188	0.215	0.138	0.302	0.241	0.272	0.241	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Government	State	0.144	0.129	0.144	0.433	0.089	0.087	0.106	0.087	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Local	0.072	0.063	0.072	0.177	0.057	0.188	0.067	0.188	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Republican	0.100	0.090	0.100	0.038	0.283	0.156	0.297	0.156	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Democrat	0.100	0.090	0.100	0.036	0.053	0.053	0.058	0.053	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Global Warming	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Envirmentalists	Pollution	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Media	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Power	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Price	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Consumers	Luxury	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Safety	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Quality	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Operating Costs	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Growth	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Macro-Economy	Employment	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Competitive	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Standard of Living	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Battery	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Technology	Range	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Engine	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Solar	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Gasoline	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Methanol	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Feul Alternatives	Electric	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Methane	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Hydrogen	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Solar	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Example 2 Sports predictions [2,4]

In addition to business applications, AHP can be used to evaluate the performance of sporting events. For instance, AHP has been used to predict which NBA (National Basketball Association) team would win the championship in 1991. In order to construct an accurate model the following influencing factors are considered: offense, defense, team statistics, and others (i.e. injuries, home court advantage and star players). The interaction of these components is shown in Figure 5. The supermatrix was then raised to powers to achieve a steady state. The results after normalization are shown in Table 3.

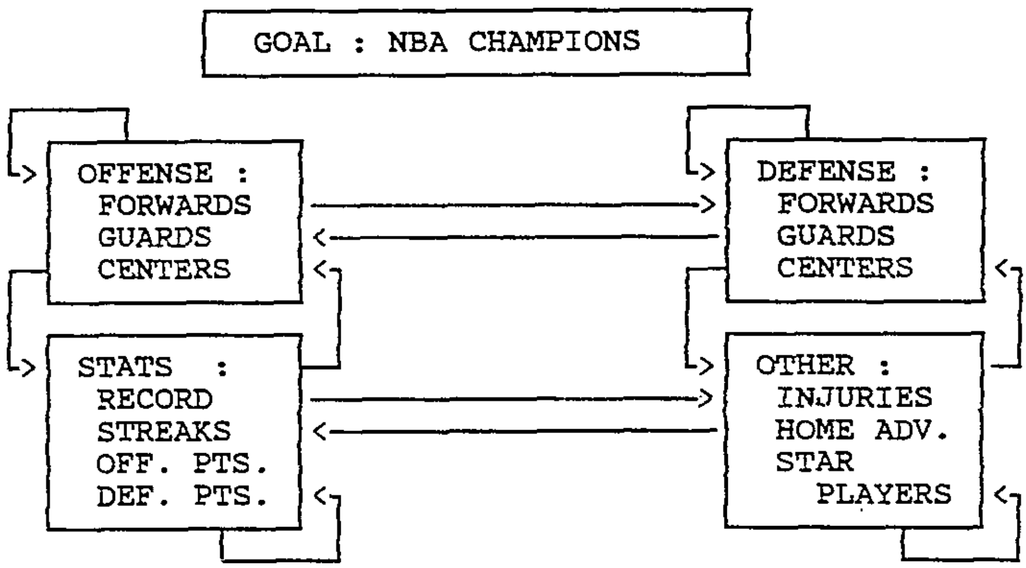

Figure 5

We need to determine how important each factor (Offense, Defense, Team Stats, Other, Teams) is relative to the others with one particular factor in mind. For example, we compare the team factors with respect to Offense. An example of our thought process for this decision proceeds as follows: a team's offensive potential influenced its offensive productivity strongly; the next most important factor affecting a team's offense was star players and injuries (Other factors); following this, a team was influenced by its record and current streak; lastly, a teams' offense was influenced by its defense. The numbers representing these factors are used in the paired comparisons.

The supermatrix is multiplied by the weighting matrix to obtain a stochastic supermatrix. This matrix is then raised to powers to obtain the following steady state result:

Table 3

Chicago	0.305
Los Angeles	0.262
Portland	0.250
Detroit	0.183

Another sporting example is the Hockey Stanley Cup Playoffs of 1991, which were the last series of hockey games from which the national champion is determined.

This problem can be approached in four ways. It can be structured as a hierarchy, a holarchy, a straight network or a network/hierarchy combination. Each of these methods indicated that the Pittsburgh Penguins would be the Stanley Cup Champions (which they were) but each was arrived at by a different route. The network model reflects the factors affecting the outcome (Figure 6) and the summary results of the four approaches is shown in Table 4.

Table 4 - AHP TECHNIQUES-SUMMARY RESULTS

	Hierarchy Eight Teams	Holarchy Without Injuries	Holarchy With Injuries	Straight Network	Network Hierarchy Combination
Pittsburgh	0.145	0.326	0.289	0.073	0.257
Edmonton	0.144	0.206	0.216	0.066	0.249
Boston	0.136	0.281	0.294	0.068	0.248
Minnesota	0.103	0.187	0.201	0.067	0.246
St. Louis	0.135				
Montreal	0.133				
LA	0.118				
Washington	0.086				

Normalized Semifinalists :

	Hierarchy	Holarchy Without Injuries	Holarchy With Injuries	Straight Network	Network Hierarchy Combination	Average
Pittsburgh	0.275	0.326	0.289	0.266	0.257	0.283
Edmonton	0.273	0.206	0.216	0.241	0.249	0.237
Boston	0.258	0.281	0.294	0.248	0.248	0.266
Minnesota	0.195	0.187	0.201	0.245	0.246	0.215

1991 STANLEY CUP PLAYOFFS NETWORK

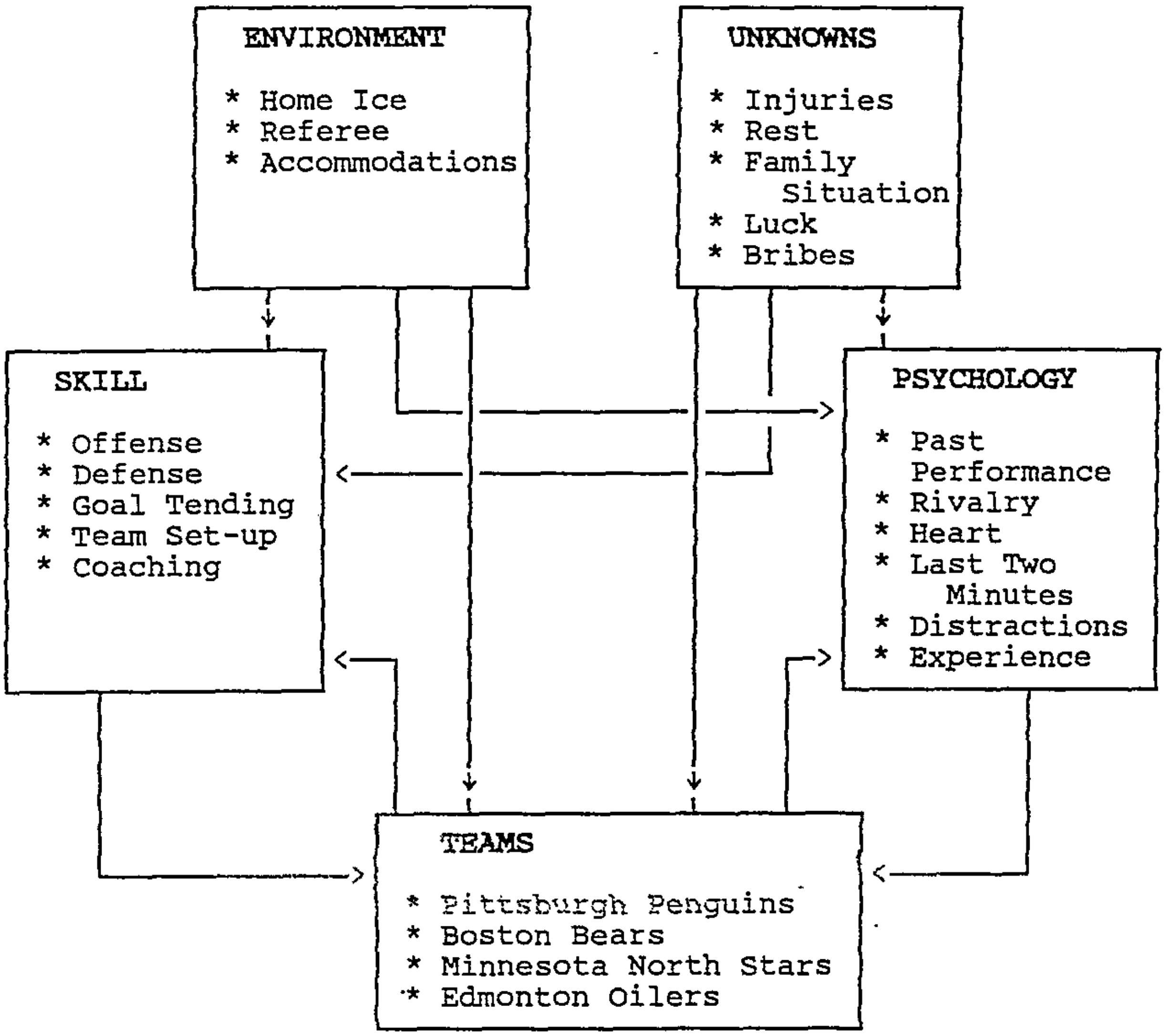

Figure 6

Example 3 Inner-Inner Dependence [3]

In a dart throwing exercise by three individuals, an attempt was made to predict the outcome, which was later validated by actually throwing darts. Six criteria were used to determine the relative dart-throwing abilities of the actors. The short term factors are: mental condition, physical condition; the long term factors are: mental skill, physical skill; and the environmental factors: visual, audio ability and influence. Only four of six criteria were compared for their contribution to each of them; the two left out belong to the category to which a criterion belongs.

For example, it was thought to be meaningless to compare mental or physical condition for their effect on mental condition. Zeros were entered in the supermatrix for these. One obtains the following supermatrix and its resulting limiting matrix whose idential columns in this case correspond to the solution of the eigenvalue problem **Ax=x**. For computational purposes it is easier to raise the matrix to powers.

	Mental Condition	Physical Condition	Mental Skill	Physical Skill	Visual Factor	Audio Factor
Mental Condition	0	0	.429	.100	.450	.445
Physical Condition	0	0	.072	.400	.050	.056
Mental Skill	.375	.072	0	0	.429	.438
Physical Skill	.125	.429	0	0	.072	.063
Visual Factor	.100	.063	.167	.072	0	0
Audio Factor	.400	.438	.334	.429	0	0

	Mental Condition	Physical Condition	Mental Skill	Physical Skill	Visual Factor	Audio Factor
Mental Condition	.276	.276	.276	.276	.276	.276
Physical Condition	.079	.079	.079	.079	.079	.079
Mental Skill	.248	.248	.248	.248	.248	.248
Physical Skill	.079	.079	.079	.079	.079	.079
Visual Factor	.069	.069	.069	.069	.069	.069
Audio Factor	.257	.257	.257	.257	.257	.257

The individuals were then compared in pairs against each of the six factors. Hierarchic composition yielded .268, .164, and .568 for their success priorities at throwing darts. Their relative score for 130 throws by each yielded respectively .26, .23 and .51. Those who did the exercise, attributed the lack of an even higher accuracy to their inability to compare the criteria with greater confidence.

References

1. Bennett, E.J., "Alternative Fuels For Automobiles", University of Pittsburgh, 1991.

2. Duffy, T.P. and A.M. Stuart, "AHP Techniques as Tools for Prediction: The 1991 Stanley Cup Playoffs, University of Pittsburgh, 1991.

3. Hauser, D., M. Baker, and M. Ogawa, "Prediction Modeling : AHP and Darts", University of Pittsburgh, 1991.

4. Rabin, A. and T. Simpson, "Predicting the 1991 NBA Championships", University of Pittsburgh, 1991.

5. Saaty, T.L. and M. Takizawa, "Dependence and Independence: From Linear Hierarchies to Nonlinear Networks,", European Journal of Operational Research, Vol. 26, No. 2, August, 1986, pp. 229-237.

6. Saaty, T.L., "How to Handle Dependence with the Analytic Hierarchy Process," International Journal of Mathematical Modelling, Vol. 9, No. 3-5, pp. 369-376, 1987.

7. Saaty, T.L., "Priorities in Systems with Feedback," International Journal of Systems, Measurement and Decisions, Vol. 1, No.1, 1981

8. Saaty, T.L., "Rationing Energy to Industries: Priorities and Input-Output Dependence," (with Reynaldo S. Mariano), Energy Systems and Policy, Vol. 8, 1979, pp. 85-111.

9. Saaty, T.L., "Multicriteria Decisions in Systems with Feedback," in Mathematical Modeling in Science and Technology - The Fourth International Conference, Pergamon Press, New York, 1984

10. Saaty, T.L., The Analytic Hierarchy Process, RWS Publications, Pittsburgh, 1990.