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PRIORITIES DERIVED FROM PAIRWISE PREFERENCES VIA SAATY'S METHOD AND THE BRADLEY-TERRY MODEL: WHAT IS THE CONNECTION?

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Abstract: Suppose that priority weights $w_1 > 0, \ldots, w_n > 0$ with $\sum_{i=1}^{n} w_i = 1$ are sought for $n \geq 3$ alternatives to be confronted on a ratio scale with respect to a single criterion, and assume that comparisons involving all possible pairs of items yielded preference ratings $0 < r_{ij} < \infty$ with the property that $r_{ji} = 1/r_{ij}$ for all $1 \leq i, j \leq n$. If $\lambda_{\max}$ is the Perron eigenvalue of the positive square matrix $R = (r_{ij})$, Saaty's eigenvector estimate of w is given by the unique element $\hat{w}$ of the simplex that verifies the linear system $R\hat{w} = \lambda_{\max}\hat{w}$.

If r_{ij} is regarded as a subjective estimation of the ratio w_i/w_j, it may be modelled statistically as $r_{ij} = (w_i/w_j)\epsilon_{ij}$ in terms of a multiplicative perturbation $\epsilon_{ij} > 0$. When r_{ij} is a ratio, as when the respondent's judgements are expressed on the nine-point scale, an alternative is to view r_{ij} as the observed win-to-loss ratio $X_{ij}/(N - X_{ij})$ resulting from a fixed number, N, of confrontations between items i and j. In accordance with the classical Bradley-Terry model for paired comparisons (Bradley & Terry, *Biometrika*, 1954), it may be reasonable to treat the X_{ij}'s as independent binomial random variables with common parameter N and unknown success probabilities $p_{ij} = w_i/(w_i + w_j)$, $1 \leq i < j \leq n$. Maximization of the associated likelihood function then results in alternative estimates $\hat{\hat{w}}_i$ for the underlying priorities of the items.

What are the properties of this estimation procedure, and how does it relate to Saaty's solution to the scaling problem? Partial answers to these questions will be presented through a series of examples and theoretical developments involving Jensen-type response matrices, as considered by Genest, Lapointe & Drury (*Journal of Mathematical Psychology*, 1993).