ORDINALITY CONSISTENCY TEST ABOUT ITEMS AND NOTATION OF A PAIRWISE COMPARISON MATRIX IN AHP

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ABSTRACT

The primary objective of this paper is to propose a consistency test for ordinality of items in a pairwise comparison matrix in the Analytic Hierarchy Process (AHP) as in a sensory test. A pairwise comparison matrix in AHP consists of elements expressed on a numerical scale. Since we can consider a numerical scale as an ordinal scale, we can transform the pairwise comparison matrix to one expressed on an ordinal scale. Additionally, we are interested whether the hypothesis that items in a pairwise comparison matrix are ranked linearly according to the transformed one is valid or not. In 1940 Kendall and Babington Smith proposed a consistency test about ordinality of items using the number of circular triads in a preference table without ties and we have used it now in a sensory test. In this paper we show how to apply their test to a pairwise comparison matrix in AHP. Difficulties have been apparent when applying it to a pairwise comparison matrix with a tie in AHP, though it is easier to do so without ties. As a consequence, we propose a method of applying it to one with a tie in AHP. Furthermore, we have researched a method of describing a pairwise comparison matrix in which relations among items can easily be observed, for instance rough-and-ready ordinality of items. A further purpose of this paper is to propose useful notation of a pairwise comparison matrix in AHP with some conditions and enumerate an example of showing its effectiveness.

Keywords: pairwise comparison matrix, consistency test, circular triad, notation

1. Introduction

In the analytic hierarchy process (AHP) by Saaty we firstly structure a hierarchy in order to clarify a given problem. Next by pairwise comparisons between items, which are criteria or alternatives, we obtain the pairwise comparison matrix about them and calculate their relative weights by the eigenvalue method or the logarithmic least square method. Finally we have weights of alternatives for the main objective aggregating the relative weights of items.

In this paper we deal with a pairwise comparison matrix by a decision maker. The primary objective of this paper is to propose a consistency test for ordinality of items in a pairwise comparison matrix in AHP. This test is to check whether or not we can accept items which were compared pairwisely are ranked linearly by the pairwise comparison matrix. In (Iida, 2009) we showed this test for a pairwise comparison matrix without a tie between different items which is introduced briefly in Section 2.1. In Section 2.2 we propose a method in the case that a pairwise comparison matrix has a tie between different items.

Jensen and Hicks (Jensen and Hicks, 1993) researched the relationship between the number of circular triads in a pairwise comparison matrix and inconsistency of the matrix in AHP. Though the purpose of their paper is different from one of this paper, they dealt with a pairwise comparison matrix with a tie between different items in their paper.

For example we can always calculate the weights of items from any given pairwise comparison matrix in AHP. So it is convenient to check whether or not we accept that the given items are ranked linearly before estimating those weights. In AHP, in general, we use a ratio scale with integers 1 to 9 and the reciprocals of them. Because this test is a kind of ranking problem, we deal only with up to 9 items and pairwise comparison matrices for them in this paper.

Furthermore, when considering ranking problems, it is useful to describe a pairwise comparison matrix which seems to show the ranking of items intuitively. In the rest of this paper we define such notation of a pairwise comparison matrix though this description doesn't have uniqueness.

2. Ordinality consistency test about items in a pairwise comparison matrix

In this section we explain how to test whether or not we can accept that items in the pairwise comparison matrix are ranked linearly. This is an arrangement of the test used in a sensory test (cf. (Research Committee, 2002)). In AHP we use the scale $\{1/k, k \mid 1 \le k \le 9$, k is an integer$\}$ to make a pairwise comparison matrix $A = (a_{ij})$. If we have $a_{ij} > 1$, then we describe $O_i \to O_j$ and construct a directed graph A_G from A. So we can discuss the number of circular triads which are $O_i \to O_j \to O_k \to O_i$ ($i \ne j, j \ne k, k \ne i$). A circular triad was called 3-cycle in (Saaty, 1977, 1980). It is well-known that circular triads cause the inconsistency of a pairwise comparison matrix. In fact if a pairwise comparison matrix is completely consistency, then there is no circular triad in it. We notice that the probabilities of $O_i \to O_j$ and $O_i \leftarrow O_j$ are equivalent to each other when we take at random a value a_{ij} from the scale $\{1/k, k \mid 2 \le k \le 9$, k is an integer$\}$ for items O_i and O_j ($i \ne j$).

We need for this test the following tables, which have the maximum values $d_{\alpha,n}$ of the number of circular triads d for each n ($3 \le n \le 9$) satisfying that probability $\Pr[\, 0 \le x \le d \,] < \alpha$ for $\alpha = 0.05$ and 0.1, respectively (see Remark 1). These values were calculated by Kendall and Babington Smith (Kendall, and Babington Smith, 1940) when $3 \le n \le 7$ and by Alway (Alway, 1962) when $n = 8$ and 9. We use $d_{0.05,n}$ for $n \ge 6$ according to (Research Committee, 2002) in a sensory test.

Table 1. The maximum number $d_{\alpha,n}$ in the numbers of circular triads d for n items such that $\Pr[\, 0 \le x \le d \,] < \alpha$ for $\alpha = 0.05$ or 0.1

n	3	4	5	6	7	8	9
$d_{0.05,n}$	0	0	0	1	3	7	13
$d_{0.1,n}$	0	0	1	2	5	9	15

Remark 1. (1) For n=3 to 5 (for n=3 and 4) there is no the maximum number of circular triads satisfying $\Pr[\, 0 \le x \le d \,] < \alpha$ for $\alpha = 0.05$ ($\alpha = 0.1$), respectively, though we filled in 0 for these n. It is natural because the matrix is completely consistent if there is no circular triad in a pairwise comparison matrix.
(2) Though $\Pr[\, 0 \le x \le 1 \,] \approx 0.051$ for n=6, $\Pr[\, 0 \le x \le 1 \,] \approx 0.117$ for n=5, $\Pr[\, 0 \le x \le 2 \,] \approx 0.120$ for n=6 and $\Pr[\, 0 \le x \le 5 \,] \approx 0.112$ for n=7, we set $d_{0.05,6}$=1, $d_{0.1,5}$=1, $d_{0.1,6}$=2 and $d_{0.1,7}$=5 in Table 1, respectively, in consideration of the distribution of d being discrete.

In a sensory test we use Table 1 to check whether or not an observer who makes pairwise comparisons is sufficiently capable of making judgments with assumption that objects are ranking linearly. On the other hand in AHP we use Tables 1 to check whether or not items are ranked linearly under the assumption that a decision maker is sufficiently capable of making judgments. Indeed if a decision maker is not capable of making pairwise comparisons, we should not use the AHP method.

Now we explain the consistency test for ordinality of items by the pairwise comparison matrix. It is clear that if the given items in AHP are ranked linearly, the pairwise comparison matrix on the AHP's scale contains no circular triad by the right judgments of a decision maker who is capable of making judgments. On the other hand a pairwise comparison matrix may contain some deviation. So we think that we can accept that the items are ranked linearly if the number of circular triads in the pairwise comparison matrix for *n* items ($3 \le n \le 9$) have less than or equal to $d_{\alpha,n}$ for $\alpha = 0.05$ or 0.1. We divide them into two cases according to whether a pairwise comparison matrix contains a tie between different items or not.

2.1 Test for a pairwise comparison matrix without a tie between different items

According to (Iida, 2009) we explain the procedure (Step 1)–(Step 3) to test whether we can accept that items are ranked linearly or not, under the hypothesis that the decision maker is capable of ranking the given items. We suppose that a decision maker compares *n* items ($3 \le n \le 9$) pairwisely to get a pairwise comparison matrix $A = (a_{ij})$. We decide between significant levels α =0.05 and 0.1. We recommend α =0.1 because human decision making is essentially ambiguous though the traditional tests in a statistical method adopting α =0.05.

(Step 1) Count the number of integers j such that $a_{ij} > 1$ for each i-th row, which is denoted by a_i.
(Step 2) Calculate the number d of circular triads in A using the following;

$$d = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{i=1}^{n} a_i^2 .$$

(Step 3) If $d \le d_{\alpha,n}$ from Tables 1, then we can assume that items are sufficiently ranked linearly.

Otherwise we cannot do so. Then we have some solutions as follows;

(1) We refer to the following coefficient of consistency ξ for A in order to estimate directly whether items are ranked linearly or not;

$$\xi = 1 - \frac{d}{s} \text{ , where } s = \begin{cases} \dfrac{n^3 - n}{24}, & \text{if } n \text{ is odd,} \\ \dfrac{n^3 - 4n}{24} & \text{if } n \text{ is even,} \end{cases}$$

where d is the number of circular triads in A which is calculated in (Step 2) and s is the possible maximum value of the number of circular triads in pairwise comparison matrices for n items. The coefficient of consistency ξ was defined by Kendall and Babington Smith. If ξ is near to 0, then the hypothesis that items are ranked linearly should be rejected as in a sensory test. This coefficient in AHP was mentioned in (Saaty, 1971, 1980) and was researched in detail in (Jensen, 1993).
(2) We recheck some elements a_{ij} in A. For instance, a consistency improving method by graph theory is proposed in (Nishizawa, 1995). When $d \ge d_{\alpha,n} + 1$ in the end, the followings are considered (see (Kendall, and Babington Smith, 1940)).
(a) Some of the items may differ by amounts which fall below the threshold of distinguishability for the decision maker.
(b) The property under judgment may not be a linear variate at all.
(c) Several of the effects may be operating simultaneously.

Remark 2. (1) The formula in (Step 2) for the number of circular triads d in A is by Kendall and Babington Smith (Kendall, and Babington Smith, 1940).

(2) It is natural to consider that if the pairwise comparison matrix made by a decision maker has no tie between different items, the same pairwise comparison matrix is given even though he or she uses the scale $\{1/k, k \mid 2 \le k \le 9, k \text{ is an integer}\}$ instead of the scale $\{1/k, k \mid 1 \le k \le 9, k \text{ is an integer}\}$.
(3) This test should be used in order to rank some items subjectively by a decision maker though AHP is used, for instance, for sport games or matches among n teams.

Here we have the following theorem to easily check whether or not items are ranked linearly by the given pairwise comparison matrix.

Theorem 1. Let $A = (a_{ij})$ be a pairwise comparison matrix for n items ($n \ge 3$) in AHP. For each integer i such that $1 \le i \le n$, we set a_i the cardinal number of $\{a_{ij} \mid 1 \le j \le n,\ a_{ij} > 1\}$ and $S = \{a_i \mid 1 \le i \le n\}$. Then $S = \{0, 1, \ldots, n-1\}$ if and only if there exists no tie and no circular triads in A. Consequently, the items corresponding to A are completely ranked linearly.

Proof. It is easy to see.

Example 1. It follows from Theorem 1 that the following pairwise comparison table has no circular triads. In fact we have n=7 and S={0, 1, …, 6} in Theorem 1.

Table 2. Trivial example of a pairwise comparison table without a circular triad

	O_1	O_2	O_3	O_4	O_5	O_6	O_7	a_i
O_1	1	2	3	4	5	6	7	6
O_2	1/2	1	2	3	4	5	6	5
O_3	1/3	1/2	1	2	3	4	5	4
O_4	1/4	1/3	1/2	1	2	3	4	3
O_5	1/5	1/4	1/3	1/2	1	2	3	2
O_6	1/6	1/5	1/4	1/3	1/2	1	2	1
O_7	1/7	1/6	1/5	1/4	1/3	1/2	1	0

C.I.=0.033

2.2 Test for a pairwise comparison matrix with a tie between different items

For the sake of convenience, we describe $O_i = O_j$ when a_{ij}=1 ($i \ne j$) in a pairwise comparison matrix A=(a_{ij}). Then for example the triad $O_1O_2O_3$ with $O_1 \to O_2 \to O_3$ and $O_1 = O_3$, or with $O_1 \to O_2$ and $O_2 = O_3 = O_1$ is clearly inconsistent. So we need to consider these kinds of triads in addition to circular triads contained in a pairwise comparison matrix.

On the other hand we note that it is very hard to define the probabilities of $O_i \to O_j$, $O_i \leftarrow O_j$ and $O_i = O_j$, respectively, when we take at random a value from the AHP's scale. In fact if one uses the scale $\{1/k, k \mid 1 \le k \le 9, k \text{ is an integer}\}$, the probability of $O_i \to O_j$ or $O_i \leftarrow O_j$ is 8/17 and that of $O_i = O_j$ is 1/17. If one uses the scale $\{1/k, k \mid 1 \le k \le 9, k \text{ is an odd number}\}$, the probability of $O_i \to O_j$ or $O_i \leftarrow O_j$ is 4/9 and that of $O_i = O_j$ is 1/9. Furthermore, we can also consider each probability of these three cases is equal to 1/3 because these appeared equally.

For a tie between different items we can use the number from 1.1 to 1.9 for a more detailed judgment as in the group AHP (Saaty & Peniwati, 2007). However, there is for instance a possibility to which the judgment is mistaken when items may differ by amounts which fall below the threshold of distinguishability for the decision maker.

As one of solutions to these problems we propose that identity O_j with O_i when $a_{ij} = 1$, and when a pairwise comparison matrix contains ties between different items, we eliminate as many ties from A as possible before applying that test. This plan is natural because our purpose is to know whether items are ranked linearly or not. We note that in the case where there are a lot of ties in a pairwise comparison matrix, it is hard to apply this test by this way, though there are mostly only a few ties in practice even if there are.

At first we explain the procedure (Step 0) with (e_1) and (e_2) to eliminate a tie from A. We suppose that A has s ties ($1 \le s \le n$). Now when $a_{ij} = a_{ji} = 1$ ($i < j$), we eliminate this tie as follows;

(e_1) When $a_{ik} > 1$ and $a_{jk} > 1$, or $a_{ik} < 1$ and $a_{jk} < 1$ for all k except for integers k such that $a_{ik} = a_{jk} = 1$, we identify O_j with O_i in the sense of ranking problem and eliminate j-th row and j-th column out of A.

(e_2) When there is an integer k such that $a_{ik} > 1$ and $a_{jk} < 1$, or $a_{ik} < 1$ and $a_{jk} > 1$, the decision maker reevaluates a_{ij}, a_{ik} and a_{jk} in A. As result one of the following holds;

(1) The decision maker changes the value of a_{ij} to a value except for 1. As a result, the tie is eliminated from A.

(2) The decision maker changes the values of a_{ik} or a_{jk} to satisfy $a_{ik} > 1$ (< 1) and $a_{jk} > 1$ (< 1) for all k except for integers k such that $a_{ik} = a_{jk} = 1$. In this case we apply (e_1) to this new matrix.

(3) Otherwise, i.e., $a_{ik} > 1$ (< 1) and $a_{jk} < 1$ (> 1) for some integer k, we consider the two items O_i and O_j as the almost same items as each other in the sense of ranking problem and eliminate the i-th row and the i-th column, or the j-th row and the j-th column from A by force (see Remark 3(1)).

After repeating these arrangements (e_1) and (e_2) for s ties we will obtain a pairwise comparison matrix without a tie. Here we apply the procedure (Step 1)–(Step 3) in Section 2.1 to this new matrix without a tie between different items. Certainly if the type of the matrix is 1×1 or 2×2, we don't need that test at all, i.e., we can think that A is sufficiently consistent.

Remark 3. (1) In Case (e_2) if (3) holds and $a_{ik} > 1$ (< 1) and $a_{jk} < 1$ (> 1), then we need to examine the both of the case of eliminating O_i, in which we consider $a_{ik} = a_{jk} < 1$ (> 1), and the case of eliminating O_j, in which we consider $a_{ik} = a_{jk} > 1$ (< 1). We may eliminate the item that passes that test. When there is no such item, we should suggest the decision maker to reconsider items on the upper level or not to use AHP at this point because the items aren't clearly ranked linearly. From these views we note that a decision maker should be sufficiently careful to use a tie.
(2) When there exists an item with two ties or more, which is for instance like an item O_2 with $O_1 = O_2 = O_3$ and $O_3 \to O_1$, we need to investigate these items simultaneously. Then notation of a pairwise comparison matrix proposed in the next section is helpful (see Example 4).

Table 3. The pairwise comparison table for eight items

	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8
O_1	1	2	1/2	2	1/2	2	1/2	2
O_2	1/2	1	4	1	1/4	1	1/4	1
O_3	2	1/4	1	4	1	4	1	4
O_4	1/2	1	1/4	1	1/4	1	1/4	1
O_5	2	4	1	4	1	4	1	4
O_6	1/2	1	1/4	1	1/4	1	1/4	1
O_7	2	4	1	4	1	4	1	4
O_8	1/2	1	1/4	1	1/4	1	1/4	1

Example 2. We use Table 3 with C.I. = 0.07252 to explain the procedure (Step 0)–(Step 3). This table was introduced in (Kwiesielewicz, and Uden, 2002) as an example of a pairwise comparison matrix with acceptable C.I. but which is inconsistent because it contains many circular triads.

Firstly, we eliminate ties from Table 3, which is denoted by $A=(a_{ij})$, by (e_1) and (e_2). Indeed the estimations in O_5 and O_7 completely accord to each other. The estimations in O_4, O_6 and O_8 also completely accord to each other. So we construct a new pairwise comparison table for five items O_1, O_2, O_3, O_4 and O_5 with a tie as follows;

Table 4. The pairwise comparison table eliminated copies from Table 3

	O_1	O_2	O_3	O_4	O_5
O_1	1	2	1/2	2	1/2
O_2	1/2	1	4	1	1/4
O_3	2	1/4	1	4	1
O_4	1/2	1	1/4	1	1/4
O_5	2	4	1	4	1

Because $a_{24}=a_{42}=1$, $a_{21}=a_{41}<1$ but $a_{23}>1$ and $a_{43}<1$ in Table 4, in (e2) we suggest the decision maker to reevaluate an importance between O_2 and O_3 or between O_3 and O_4. For example if he or she changes $a_{23}=4$ to 1/4 (= a_{43}), then we have the following table;

Table 5. The pairwise comparison table changed a_{23} to 1/4 in Table 4

	O_1	O_2	O_3	O_4	O_5
O_1	1	2	1/2	2	1/2
O_2	1/2	1	1/4	1	1/4
O_3	2	4	1	4	1
O_4	1/2	1	1/4	1	1/4
O_5	2	4	1	4	1

Then we can identify O_2 with O_4 in Table 5 in the second times of (e_1) and so we have Table 6 by removing O_2 from Table 5. Because changing the value a_{23} of O_2, we take O_2 off Table 5 to get Table 6.

Table 6. The pairwise comparison table eliminated O_2 from Table 5

	O_1	O_3	O_4	O_5
O_1	1	1/2	2	1/2
O_3	2	1	4	1
O_4	1/2	1/4	1	1/4
O_5	2	1	4	1

As a result, we can identify O_3 with O_5 in Table 6 and have the following Table 7 in the third times of (e_1).

Table 7. The pairwise comparison table eliminated completely ties from Table 3

	O_1	O_3	O_4
O_1	1	1/2	2
O_3	2	1	4
O_4	1/2	1/4	1

Here we apply the test proposed in Section 2.1 to Table 7, which is called $A=(a_{ij})$ from now, in order to explain (Step 1)–(Step 3), though it follows from Theorem 1 that there are no circular triads in Table 7. We decide a significant level $\alpha = 0.1$. In (Step 1) we have the following table about a_i in Table 8.

Table 8. The number a_i of integers *j* such that $a_{ij} > 1$

a_1	a_2	a_3
1	2	0

In (Step 2) we have d=0 from Table 8. In (Step 3) because $d=0 \le d_{0.1,3}=0$ by Table 1, we can assume that these items are ranked linearly by the pairwise comparison matrix.

On the other hand, it is easy to see that if the decision maker changes a_{43}=1/4 to 4 (= a_{23}) in Table 4, then Table 5 cannot pass this test. Consequently, we need to change a_{23}=4 and a_{32}=1/4 to 1/4 and 4 in Table 3, respectively, in order to pass this test.

3. Notation of a pairwise comparison matrix

In this section we propose notation of a pairwise comparison matrix in which relations among items can easily be observed, for instance rough-and-ready ordinality of items. It is useful to know the ranking of items simply by the number of wins and the number of defeats on the other hand of ranking by weights of AHP. In fact the former is fit for the decision maker's intuition. We propose the standardization of two stages. Though this notation doesn't have uniqueness, we show the utility of it by an example at the end of this section.

Now we consider the relation between items and the pairwise comparison table for it. For example, we describe the pairwise comparison table $A = (a_{ij})$ for 5 items O_i ($1 \le i \le 5$) as follows;

Table 9. A pairwise comparison matrix $A = (a_{ij})$ for $\{O_1, O_2, O_3, O_4, O_5\}$

	O_1	O_2	O_3	O_4	O_5
O_1	1	a_{12}	a_{13}	a_{14}	a_{15}
O_2	a_{21}	1	a_{23}	a_{24}	a_{25}
O_3	a_{31}	a_{32}	1	a_{34}	a_{35}
O_4	a_{41}	a_{42}	a_{43}	1	a_{45}
O_5	a_{51}	a_{52}	a_{53}	a_{54}	1

Then the *i*-th row and the *i*-th column of *A* are corresponding to the item O_i. In general an ordinality of items corresponding to rows and columns of a pairwise comparison matrix is decided by a decision maker without a rule. So when representing pairwise comparisons with a table, we need to show items corresponding to rows and columns of the table, for example, the pairwise comparison table *A* for $\{O_1, O_2, O_3, O_4, O_5\}$. In general, the table for $\{O_1, O_4, O_3, O_2, O_5\}$ is different from one for $\{O_1, O_2, O_3, O_4, O_5\}$. By the way, in order to obtain a pairwise comparison table for $\{O_1, O_4, O_3, O_2, O_5\}$ from one for $\{O_1, O_2, O_3, O_4, O_5\}$, it is sufficient to exchange the 2-th column for the 4-th column after exchanging the 2-th row for the 4-th row as follows. Table 11 is the pairwise comparison matrix for $\{O_1, O_4, O_3, O_2, O_5\}$.

Table 10. The table which is exchanged the 2-th row for the 4-th row in Table 9

	O_1	O_2	O_3	O_4	O_5
O_1	1	a_{12}	a_{13}	a_{14}	a_{15}
O_4	a_{41}	a_{42}	a_{43}	1	a_{45}
O_3	a_{31}	a_{32}	1	a_{34}	a_{35}
O_2	a_{21}	1	a_{23}	a_{24}	a_{25}
O_5	a_{51}	a_{52}	a_{53}	a_{54}	1

Table 11. The table which is exchanged the 2-th column for the 4-th column in Table 10

	O_1	O_4	O_3	O_2	O_5
O_1	1	a_{14}	a_{13}	a_{12}	a_{15}
O_4	a_{41}	1	a_{43}	a_{42}	a_{45}
O_3	a_{31}	a_{34}	1	a_{32}	a_{35}
O_2	a_{21}	a_{24}	a_{23}	1	a_{25}
O_5	a_{51}	a_{54}	a_{53}	a_{52}	1

Here we rewrite this method in a matrix. We denote by E(i, j) the matrix which is exchanged the i-row for the j-row in the identity matrix. For example, in order to exchange the 2-th row for the 4-th row in a pairwise comparison matrix A for 5 items we use the following matrix;

$$\mathrm{E}(2,4)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then it is easy to see that multiplying E(2,4) from the right (left) side by a matrix we have the matrix that the 2-th row (column) and the 4-th row (column) of the matrix are exchanged. Consequently, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & 1 & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 1 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & 1 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & a_{14} & a_{13} & a_{12} & a_{15} \\ a_{41} & 1 & a_{43} & a_{42} & a_{45} \\ a_{31} & a_{34} & 1 & a_{32} & a_{35} \\ a_{21} & a_{24} & a_{23} & 1 & a_{25} \\ a_{51} & a_{54} & a_{53} & a_{52} & 1 \end{pmatrix}.$$

In general, we obtain the pairwise comparison matrix B for $\{O_{k(1)}, O_{k(2)}, O_{k(3)}, O_{k(4)}, O_{k(5)}\}$ from the pairwise comparison matrix A for $\{O_1, O_2, O_3, O_4, O_5\}$ as follows. We have the following permutation σ in the symmetric group S_5 for 5 letters;

$$\sigma=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ k(1) & k(2) & k(3) & k(4) & k(5) \end{pmatrix}\in S_5.$$

From elementary algebra it follows that there exist transpositions (i_1, j_1), (i_2, j_2), …, and (i_t, j_t) for some t such that $\sigma = (i_1, j_1)(i_2, j_2)\ldots(i_t, j_t)$. Then $B = \mathrm{E}(i_t, j_t)\ldots\mathrm{E}(i_2, j_2)\mathrm{E}(i_1, j_1)\, A\, \mathrm{E}(i_1, j_1)\mathrm{E}(i_2, j_2)\ldots\mathrm{E}(i_t, j_t)$.

On the other hand the following is well-known.

Lemma 1. Let A and B be square $n \times n$ matrices. If there exists a non-singular $n \times n$ matrix C such that $B=C^{-1}AC$, the followings hold.
(1) Eigenvalues of A are equivalent to eigenvalues of B.
(2) If x is an eigenvector of A belonging to the eigenvalue λ of A, then $C^{-1}x$ is an eigenvector of B belonging to the eigenvalue λ of B. On the other hand, if y is an eigenvector of B belonging to the eigenvalue λ of B, then Cy is an eigenvector of A belonging to the eigenvalue λ of A.

From the above argument and Lemma 1 we have the following, which is a base of the eigenvalue method in AHP.

Theorem 2. Weights of items, which were compared pairwisely, calculated by the eigenvalue method in AHP are independent upon how to make a pairwise comparison matrix for these items.

Proof. It is easily proved from Lemma 1 since $E(i, j)^{-1} = E(i, j)$.

Now we define the standardization of two stages for a pairwise comparison matrix $A = (a_{ij})$.

(First Standardization) Arrange items from the first row in lexicographic order of values of $w_i = \#\{a_{ij} \mid a_{ij} > 1\}$ and $t_i = \#\{a_{ij} \mid a_{ij} = 1\}$ $(1 \le i \le n)$ in $A = (a_{ij})$, where we denote by #S the cardinal number of a set S. Consequently, we have a first standardized matrix $A' = (a'_{ij})$.

(Second Standardization) Moreover, we standardize a first standardized matrix $A' = (a'_{ij})$. If there exist items having the same values w_i and t_i in $A' = (a'_{ij})$, arrange those items from the upper row in descending order of $g_i = \prod_{j=1}^{n} a_{ij}$. Consequently, we have a second standardized matrix $A'' = (a''_{ij})$.

We note that weights of items by the eigenvalue method are invariant by the above two standardizations. From Second standardization we know whether or not there exists a k-cycle ($k \ge 3$) in the pairwise comparison matrix, and if the k-cycle exists in complete form, i.e., all items in k-cycle have the same relations with any items in A'' except for themselves or not, and so on. These standardizations are useful when analyzing a single pairwise comparison matrix, though when analyzing or comparing more than one pairwise comparison matrices for alternatives under several criteria, these are not necessarily useful for a decision maker.

We show how to standardize a pairwise comparison matrix by an example.

Example 3. We suppose that a decision maker pairwisely compared 6 items, which are denoted by A, B, C, D, E and F and made the following pairwise comparison table $P = (p_{ij})$;

Table 12. The pairwise comparison table P for { A, B, C, D, E, F}

	A	B	C	D	E	F
A	1	5	4	7	1	2
B	1/5	1	1	1/2	2	1/2
C	1/4	1	1	2	1/2	2
D	1/7	2	1/2	1	1/4	1/4
E	1	1/2	2	4	1	1
F	1/2	2	1/2	4	1	1

We firstly calculate w_i $(1 \le i \le 5)$ which is the cardinal number of $\{ p_{ij} \mid p_{ij} > 1\}$ and t_i $(1 \le i \le 5)$ which is the cardinal number of $\{ p_{ij} \mid p_{ij} = 1\}$, and exchange the items from the first row in lexicographic order of values of w_i and t_i $(1 \le i \le 5)$.

Table 13. The table which is applied First Standardization in Table 12

	A	E	C	F	B	D	w_i	t_i
A	1	1	4	2	5	7	4	2
E	1	1	2	1	1/2	4	2	3
C	1/4	1/2	1	2	1	2	2	2
F	1/2	1	1/2	1	2	4	2	2
B	1/5	2	1	1/2	1	1/2	1	2
D	1/7	1/4	1/2	1/4	2	1	1	1

Here we notice that items *C* and *F* in Table 13 have the same w_i and t_i. So we calculate $g_i = \prod_{j=1}^{n} p_{ij}$ for *C* and *F*, respectively, and arrange these two items from the upper row in descending order of g_i.

Table 14. The table which is applied Second Standardization in Table 13

	A	E	F	C	B	D	g_i
A	1	1	2	4	5	7	
E	1	1	1	2	1/2	4	
F	1/2	1	1	1/2	2	4	2
C	1/4	1/2	2	1	1	2	1/2
B	1/5	2	1/2	1	1	1/2	
D	1/7	1/4	1/4	1/2	2	1	

Finally we change the names of items A, E, F, C, B and D to O_1, O_2, O_3, O_4, O_5 and O_6, respectively and obtain the following table which is applied Second Standardization.

Table 15. The pairwise comparison matrix with Second Standardization for $\{O_1, O_2, O_3, O_4, O_5, O_6\}$

	O_1	O_2	O_3	O_4	O_5	O_6
O_1	1	1	2	4	5	7
O_2	1	1	1	2	1/2	4
O_3	1/2	1	1	1/2	2	4
O_4	1/4	1/2	2	1	1	2
O_5	1/5	2	1/2	1	1	1/2
O_6	1/7	1/4	1/4	1/2	2	1

Here we note that the both Standardization are independent on each other. Indeed there is an example in which O_1 is arranged above O_2 by First Standardization even though $g_1<g_2$ as follows;

Table 16. A standardized table with $g_1<g_2$

	O_1	O_2	O_3	w_i	t_i	g_i
O_1	1	2	2	2	1	4
O_2	1/2	1	9	1	1	9/2 = 4.5
O_3	1/2	1/9	1	0	1	1/18

In the next table, we cannot uniquely arrange items O_1, O_2 and O_3 by these two standardizations and 3-cycle of these items isn't complete.

Table 17. A table which doesn't have the unique form by two Standardizations

	O_1	O_2	O_3	O_4	O_5	w_i	t_i	g_i
O_1	1	3	1/3	5	4	3	1	20
O_2	1/3	1	3	5	4	3	1	20
O_3	3	1/3	1	4	5	3	1	20
O_4	1/5	1/5	1/4	1	2	1	1	
O_5	1/4	1/4	1/5	1/2	1	0	1	

Finally, we show that these standardizations are useful for decision makers by an example. In addition, we apply the consistency test introduced in Section 2 to this example again (see Example 2).

Example 4. We deal with the pairwise comparison matrix (table) A by Kwiesielewicz and Uden in Example 2.

$$A = \begin{pmatrix} 1 & 2 & 1/2 & 2 & 1/2 & 2 & 1/2 & 2 \\ 1/2 & 1 & 4 & 1 & 1/4 & 1 & 1/4 & 1 \\ 2 & 1/4 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 2 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 2 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 2 & 1/4 & 4 & 4 & 4 \\ 1/2 & 1/2 & 1/2 & 1 & 2 & 2 & 2 & 2 \\ 1/4 & 1/4 & 4 & 1/2 & 1 & 1 & 1 & 1 \\ 1/4 & 1/4 & 1/4 & 1/2 & 1 & 1 & 1 & 1 \\ 1/4 & 1/4 & 1/4 & 1/2 & 1 & 1 & 1 & 1 \\ 1/4 & 1/4 & 1/4 & 1/2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Then we have the matrix B by First and Second Standardizations. It is very easy to see that there are about three kinds of items in B, i.e., A, in the sense of ranking problem. When setting items corresponding to $B = (b_{ij})$ O_1, O_2, …, and O_8 from the first row of B, it easily follows that b_{35} and b_{53} have problems. So we can advise a decision maker to recheck the comparison for O_3 and O_5.

Here we apply our test for ordinality of items according to (Step 0)–(Step 3) in Section 2. For simplicity we use the matrix B instead of the original matrix A for this test. From procedure (e_1) we have the following matrix $C_0 = (c_{ij})$ by identifying O_2 with O_1, and doing O_7 and O_8 with O_6. We note that items in C_0 are O_1, O_3, O_4, O_5 and O_6 from the top of rows.

$$C_0 = \begin{pmatrix} 1 & 1 & 2 & 4 & 4 \\ 1 & 1 & 2 & 1/4 & 4 \\ 1/2 & 1/2 & 1 & 2 & 2 \\ 1/4 & 4 & 1/2 & 1 & 1 \\ 1/4 & 1/4 & 1/2 & 1 & 1 \end{pmatrix}$$

In procedure (e_2) because of $c_{12}=c_{21}=1$, we make an effort to identify O_3 with O_1 in the sense of ranking problem. If, for example, we set $c_{24}=4$, then we can identify O_3 with O_1, and O_6 with O_5 by the second procedure (e_1) and have the following matrix C_1. It follows from Theorem 1 that C_1 have no circular triad and so we know that the pairwise comparison matrix B, i.e., A, passes the test and items in A are ranked linearly by A. On the other hand, if we set $c_{14}=1/4$, then we have the following matrix C_2 in the same procedure. It is easy to see that C_2 has a circular triad and so the matrix B, i.e., A, doesn't pass the test by Table 1 because we have $d=1$ in (Step 2).

$$C_1 = \begin{pmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{pmatrix}, \; C_2 = \begin{pmatrix} 1 & 2 & 1/4 \\ 1/2 & 1 & 2 \\ 4 & 1/2 & 1 \end{pmatrix}$$

Consequently, we need to ask the decision maker if the element c_{24} in C_0, which a_{32} is in A, might be changed 4 or at least an integer more than 1. This result is the same as one in Example 2.

4. Conclusions

The purpose of AHP is to rank some items linearly. So it is useful for a decision maker to know whether items which are compared pairwisely could be ranked linearly by the pairwise comparison matrix and we proposed two methods in this paper. One is an ordinary consistency test introduced in Section 2 and another is notation defined in Section 3. We showed in Example 4 that there is some relation between these methods. We think that these two methods make AHP more useful and helpful, in particular for beginners who aren't familiar with AHP.

Finally, we couldn't clarify the detailed relation between ranking by the numbers of wins and defeats and ranking by the eigenvalue method in AHP. After calculating weights of items by the eigenvalue method, it is useful for a decision maker to compare these two rankings and if they are different, then we might need to check some elements in the pairwise comparison matrix. These are topics for further research.

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