SOLUTIONS FOR A DISJOINT SUPERMATRIX IN ANP DECISION MODELS

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ABSTRACT

Obtaining meaningful priority vectors from an Analytic Network Process (ANP) decision model requires the adherence to the principles and axioms of the ANP. The alternatives and criteria being considered must be strongly connected in order to obtain a meaningful priority vector. A simple yet delectable example is presented to demonstrate the issues that can arise when a decision maker forms a disjoint Supermatrix and that the frequency of such decisions occurring in practice can be very common. From the example it can be observed that the necessary information to complete additional linking comparisons already exists outside the Supermatrix; and by performing linking comparisons a decision maker can convert a disjoint Supermatrix into a strongly connected Supermatrix. The linking process is summarized in five steps and generalized mathematically. This linking comparison methodology can also be used to weight the criteria clusters within a network by making pairwise comparisons at the level of a criterion of a single alternative with respect to another criterion of that same alternative. Performing the comparisons at the level of the alternatives can reduce the decision maker's cognitive burden and allow for more redundancy in cluster comparisons to increase a decision makers' overall consistency. The ability to strongly connect an otherwise disjoint Supermatrix and reduce the decision maker's cognitive burden demonstrates the usefulness of linking comparisons.

Keywords: Decision analysis, ANP, Disjoint matrix, Linking comparisons

1. Introduction

The use of the Analytic Hierarchy/Network Process (AHP/ANP) as a Multi-Criterion Decision Making (MCDM) tool has continued to find additional application within the field of operations research (OR). In a literature review of OR decision making articles published in between 1970-2007, Wallenius, et al. (2008) report how the use of MCDM tools has shifted and the use of ANP has increased and surpassed the use of other decision making methods in journals listed in the ISI Database. The theorems and axioms of AHP/ANP were developed by Thomas Saaty (Saaty, 1977, 1990, 2005; Saaty & Vargas, 2007) who

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combined principles from mathematics and psychology into a comprehensive decision making process capable of measuring and combining tangible and intangible criteria.
A theorem of the AHP defines a matrix of pairwise comparisons as strongly connected "if and only if every arc belongs to at least one cycle" (Saaty, 1994, p. 234) . If the pairwise comparison matrix is not strongly connected it will not converge to an eigenvector of the form

$$a_i / \sum_{i=1}^{n} a_i \qquad (1)$$

in the limit matrix, which provides the relative priority or contribution of an alternative a_i with respect to the system of *n* alternatives being considered. Sidestepping this theorem in the Supermatrix can lead to incorrect synthesized results. While the model will still converge to an answer, the results can lead to the unintended ranking of alternatives.

In order to obtain meaningful priority vectors from an Analytic Network Process (ANP) decision model the Supermatrix must be strongly connected. Put another way, the alternatives and criteria being considered in the decision must be strongly connected. While the theorem regarding strongly connected matrices is mathematically necessary, it also excludes a number of meaningful decisions from being evaluated under the ANP. A disjoint Supermatrix is not likely to occur in large networks in complex models; however in BOCR (Benefits, Opportunities, Costs, and Risks) models in particular, which can contain multiple smaller subnetworks, the likelihood increases of encountering networks that are not strongly connected. By performing a few additional pairwise comparisons a disjoint or weakly connected Supermatrix can be converted into a strongly connected Supermatrix and satisfy the theorems of the ANP. These additional comparisons are as simple to perform as regular pairwise comparisons and open the possibility to apply the ANP in even more decisions than was previously possible.

A simple yet delectable real life example is presented to demonstrate the issues that can arise when a in a disjoint Supermatrix and that the frequency of such decisions occurring in practice can be common. From the example it can be observed that the necessary information to complete the additional pairwise comparisons already exists and can be used to make the additional needed comparisons. By performing the additional comparisons the disjoint Supermatrix can be converted into a strongly connected Supermatrix. The necessary steps are summarized as a step by step process. After reviewing the relevant literature, an "Energy Model" with a disjoint Supermatrix is presented. The proposed linking comparisons solution is applied to the Energy Model example. Finally the results are summarized with some concluding remarks.

2. Literature review

A detailed presentation of the theory, principles and axioms of the ANP can be found in the following books and articles (Saaty, 1980, 1990, 1994, 2005). The concept of linking pin comparisons was developed by Schoner, et al. (1993) and is reviewed with an emphasis on its particular application within this paper; the application of linking pin comparisons herein hinges on defining the unit in multi-criteria ratios (Wedley & Choo, 2011). Each of these concepts is discussed in greater detail below.

Saaty (1980) suggests using a form of linking or "pivot" comparisons among non-homogeneous items in his classic watermelon and cherry tomato example where the relationships between the object's sizes exceed the use of the 1-9 scale. By clustering the objects within smaller clusters and putting a copy of one item from another cluster in the subsequent cluster a link or pivot is created that can be used to link the cherry tomato to the watermelon.

Schoner, et al. (1993) propose a linking pin approach in an effort to address and unify approaches to AHP and specifically addressing the principal of independence of irrelevant alternatives. The linking pin approach is a normalization process that can be used in both a hierarchy and network. This approach is particularly helpful when dealing with the addition of new alternatives and the impact on criterion weights. In this paper another important conclusion is the basis used to justify the use of linking comparisons to link the disjointed portions of the Supermatrix. Schoner, et al. (1993) demonstrate that the criteria and alternatives are structurally dependent on one another. From this dependency it can be seen that everything is related; and therefore a decision maker may “arbitrarily” select which entries within a Supermatrix to use as the linking pin comparisons to normalize the columns in the Supermatrix. The concept that everything is related is necessary when selecting which elements to compare to connect the disjoint subnetworks in the Supermatrix. The term linking comparison will be used throughout this paper to refer to the comparisons that will be used to strongly connect the disjoint subnetworks within a decision model.

Criteria weights in general are misunderstood and misused (Choo, Schoner, & Wedley, 1999). Choo, et al. demonstrate that there is no consensus on the meaning or manner of deriving criteria weights. Furthermore the criteria weights should not be calculated in a way that is independent of how they are used in a decision model. While criteria weights can be used for the normalization process, normalization in and of itself does not remove the units from the criteria being considered. According to Saaty (2004), relative scales do not need a unit of measurement. However, Wedley & Choo (2011), explain that ratio scales in the ANP have a unit of measure and the unit of measure is important and useful. The unit of measurement is derived from the topmost node in the total network. The scale that one can obtain from such a unit is transient depending on the alternatives being considered but so is the ratio scale itself. Focusing on the ratios rather than the rank will improve the efficacy of the ANP. Wedley & Choo (2011 p. 170), conclude “therein lie both the advantage and dilemma of AHP. We do not need explicit knowledge of the underlying unit of measure to derive a ratio scale, yet the derived scale has a unit.” This understanding that the unit of measurement is derived from the topmost node in the network provides a unit to use as the basis for comparing criteria across clusters.

In the next section an “Energy Model” is presented to measure which alternative has the most energy measured in calories. The design of the network emphasizes that the unit of measurement is the topmost node in the network. This unit of measurement allows for the use of linking comparisons to connect the disjoint matrices in the decision model and to perform simpler pairwise comparisons.

3. Energy model

In order to provide a model that is straightforward and easy to understand, the first example is a simple yet delectable example of choosing which of four crepes (alternatives) from a menu contains the greatest amount of energy (calories). The analogy of a recipe is particularly useful because whether a decision is a policy decision or vendor selection problem, the decision maker can break the decision down into the parts or ingredients (criteria) that determine the value of each alternative. The decision maker will determine the contribution or influence of each of the criterion on the alternatives and also how the criteria are distributed within the alternatives. These influences are captured through the pairwise comparisons and synthesized within the Supermatrix which converges to the limit matrix. The resulting eigenvector in the limit matrix provides the relative contributions of the alternatives and of each criterion.

To determine which alternative has the most energy assume there is a menu limited to four crepe selections. Each crepe (alternative) comes with a predetermined combination of Toppings (Nutella, Nutella and Banana, Powdered Sugar, or Berries and Powdered Sugar). Finally, each alternative comes with a Drink (Milk, Cocoa, or Juice) see Table 1. To keep the model simple and not distract from the intended purposes, let us further assume: first, that the menu is limited to only four alternatives; second, the menu has a fixed price for any of the four alternatives regardless of which one is ordered. This leads to the definition of the preferred crepe as the one which has the most net worth as a function of the calories in each alternative.

While in reality the relationship between the quality or "richness" of a food and the amount of calories contained therein may not hold, in this example it provides a tangible and quantifiable model that is easy to interpret and demonstrates the issues that arise when a decision matrix is disjoint and also underscores the benefits a decision maker will realize from making linking comparisons. Below is a table of the four alternatives with their respective Toppings, Drinks, total calories, and relative calories (relative weights).

Criteria	Alt 1	Alt 2	Alt 3	Alt 4	Units
Nutella	2	2	0	0	2 tablespoons
Banana	0	1	0	0	half banana
Berries	0	0	0	2	1/2 cup
Powder Sugar	0	0	1	1	1/8 cup
Milk	1	1	0	0	1 cup
Cocoa	0	0	1	0	10 oz
Juice	0	0	0	1	12 oz
Total	523.00	654.60	296.25	336.00	Calories
Relative Weights	0.28897	0.36169	0.16369	0.18565	

Table 1. Data for Crepe Energy Example

Carbohydrates

Criteria	Alt 1	Alt 2	Alt 3	Alt 4
Nutella	44	44	0	0
Banana	0	31	0	0
Berries	0	0	0	17
Powder Sugar	0	0	15	15
Milk	12	12	0	0
Cocoa	0	0	34	0
Juice	0	0	0	51
Total	56	87	49	83
Relative Weights	0.204	0.316	0.178	0.302

Protein

Criteria	Alt 1	Alt 2	Alt 3	Alt 4
Nutella	6	6	0	0
Banana	0	1	0	0
Berries	0	0	0	1
Powder Sugar	0	0	0	0
Milk	8	8	0	0
Cocoa	0	0	11	0
Juice	0	0	0	0
Total	14	15	11	1
Relative Weights	0.341	0.366	0.268	0.004

Fat

Criteria	Alt 1	Alt 2	Alt 3	Alt 4
Nutella	22	22	0	0
Banana	0	0.4	0	0
Berries	0	0	0	0
Powder Sugar	0	0	0	0
Milk	5	5	0	0
Cocoa	0	0	6.25	0
Juice	0	0	0	0
Total	27.00	27.40	6.25	0.00
Relative Weights	0.445	0.452	0.103	0.000

Table 2. Data for Comparisons

Unlike with most decisions, a decision maker in this situation has complete access to all the information provided in Table 1. In general, if such were the case where the actual total calories are known with certainty the decision would be simple and there is no need to perform the pairwise comparisons. Nonetheless, this information is put into the decision model to see if the same results are produced with the pairwise comparisons and to provide results that can be precisely interpreted. In a subsequent section, the generalization to intangibles will be shown. The pairwise comparisons in this model were made solely with respect to the data in Table 1 and the calories in each item; hence there is no inconsistency in the comparisons. While it may be difficult for decision makers to know the total amount of energy in each item the decision can be further simplified by creating additional networks. Under the goal of determining which alternative has the most energy, "energy" can be further broken down into three subcriteria: Carbohydrates, Fat, and Protein (see Figure 1). Under each subnetwork the alternatives are connected to the ingredients (criteria) they contain (see Figure 1).

First, the influence of each ingredient (criterion) with respect to each alternative was compared. For example; In Alt 2 which has more Carbohydrates, the Nutella or the Banana? How much more? These questions would be repeated for each relevant item in the Toppings cluster and the Drinks cluster.

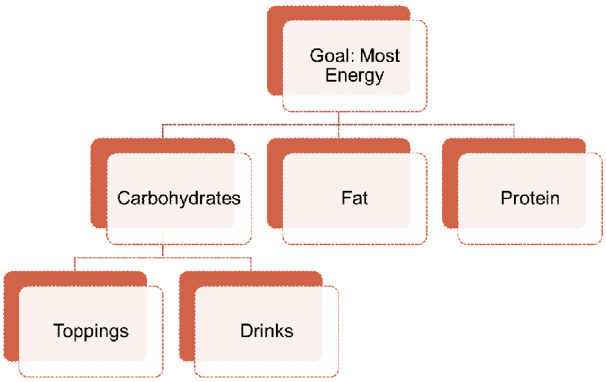

Figure 1 Crepe Energy Network with Subnetworks and Energy Subnetwork

The second set of relationships to be evaluated is of the alternatives with respect to each ingredient (criterion). For example one would ask which alternative contributes more Carbohydrates from the Nutella, Alt 1 or Alt 2? How much more? The resulting eigenvectors are entered into the unweighted Supermatrix. Both sets of comparisons should be completed within each subnetwork (Carbohydrates, Fat, and Protein) before the entire model is synthesized. The results from the Carbohydrates subnetwork will be presented below to highlight the issues that result from disjoint decision models. The unweighted Supermatrix is displayed in Table 2. The weighted Supermatrix is displayed in Table 3.

Unweighted Carbohydrates Supermatrix

		Toppings				Drinks			Alts			
		Nutella	Banana	Berries	Powder Sugar	Milk	Cocoa	Juice	Alt 1	Alt 2	Alt 3	Alt 4
Toppings	Nutella	0	0	0	0	0	0	0	1	0.587	0	0
	Banana	0	0	0	0	0	0	0	0	0.413	0	0
	Berries	0	0	0	0	0	0	0	0	0	0	0.531
	Powder Sugar	0	0	0	0	0	0	0	0	0	1	0.469
Drinks	Milk	0	0	0	0	0	0	0	1	1	0	0
	Cocoa	0	0	0	0	0	0	0	0	0	1	0
	Juice	0	0	0	0	0	0	0	0	0	0	1
Alts	Alt 1	0.5	0	0	0	0.5	0	0	0	0	0	0
	Alt 2	0.5	1	0	0	0.5	0	0	0	0	0	0
	Alt 3	0	0	0	0.5	0	1	0	0	0	0	0
	Alt 4	0	0	1	0.5	0	0	1	0	0	0	0

Table 3. Unweighted Supermatrix

Weighted Carbohydrates Supermatrix

		Toppings				Drinks			Alts			
		Nutella	Banana	Berries	Powder Sugar	Milk	Cocoa	Juice	Alt 1	Alt 2	Alt 3	Alt 4
Toppings	Nutella	0	0	0	0	0	0	0	**0.786**	**0.506**	**0**	**0**
	Banana	0	0	0	0	0	0	0	**0**	**0.356**	**0**	**0**
	Berries	0	0	0	0	0	0	0	**0**	**0**	**0**	**0.205**
	Powder Sugar	0	0	0	0	0	0	0	**0**	**0**	**0.306**	**0.181**
Drinks	Milk	0	0	0	0	0	0	0	**0.214**	**0.138**	**0**	**0**
	Cocoa	0	0	0	0	0	0	0	**0**	**0**	**0.694**	**0**
	Juice	0	0	0	0	0	0	0	**0**	**0**	**0**	**0.614**
Alts	Alt 1	**0.5**	**0**	**0**	**0**	**0.5**	**0**	**0**	0	0	0	0
	Alt 2	**0.5**	**1**	**0**	**0**	**0.5**	**0**	**0**	0	0	0	0
	Alt 3	**0**	**0**	**0**	**0.5**	**0**	**1**	**0**	0	0	0	0
	Alt 4	**0**	**0**	**1**	**0.5**	**0**	**0**	**1**	0	0	0	0

Table 4. Weighted Supermatrix

The submatrices representing the Alternatives compared to the Toppings and the Toppings compared to the Alternatives are by definition disjoint matrices. The submatrices representing the Alternatives compared to the Drinks and the Drinks compared to the Alternatives are also disjoint matrices. Due to the fact that the Supermatrix is disjoint the strongly connected matrix theorem is violated and in its current form this decision cannot be evaluated using the ANP. This example was chosen in part to also demonstrate the frequency at which decisions can violate this theorem. Furthermore, to make this decision strongly connected, in the strongest sense, each alternative must possess every criteria. Now imagine ordering a crepe that has every ingredient, with every Topping, and all three Drinks. While each alternative will have differing amounts of Nutella, Bananas, Berries and Powdered Sugar the alternatives have now become very homogenous; and for those who only wanted Nutella and Bananas they are out of luck. This problem is underscored even more with the Drinks cluster where every alternative should not come with some volume of every drink.

When the disjoint Supermatrix is raised to powers an error occurs. While the model converges, the results are incomplete. The error occurs because there is not enough communication among the nodes within the Toppings and Drinks clusters. However, there is an interesting pattern in the results and the current solution is quite useful. With a few simple additional pairwise comparisons meaningful results can be achieved as will be demonstrated in the next section.

Carbohydrates Limit Matrix (Reducible)

		Toppings				Drinks			Alts			
		Nutella	Banana	Berries	Powder Sugar	Milk	Cocoa	Juice	Alt 1	Alt 2	Alt 3	Alt 4
Toppings	Nutella	0	0	0	0	0	0	0	**0.615**	**0.615**	**0**	**0**
	Banana	0	0	0	0	0	0	0	**0.217**	**0.217**	**0**	**0**
	Berries	0	0	0	0	0	0	0	**0**	**0**	**0.129**	**0.129**
	Powder Sugar	0	0	0	0	0	0	0	**0**	**0**	**0.227**	**0.227**
Drinks	Milk	0	0	0	0	0	0	0	**0.168**	**0.168**	**0**	**0**
	Cocoa	0	0	0	0	0	0	0	**0**	**0**	**0.258**	**0.258**
	Juice	0	0	0	0	0	0	0	**0**	**0**	**0.386**	**0.386**
Alts	Alt 1	**0.392**	**0.392**	**0**	**0**	**0.392**	**0**	**0**	0	0	0	0
	Alt 2	**0.608**	**0.608**	**0**	**0**	**0.608**	**0**	**0**	0	0	0	0
	Alt 3	**0**	**0**	**0.371**	**0.371**	**0**	**0.371**	**0.371**	0	0	0	0
	Alt 4	**0**	**0**	**0.629**	**0.629**	**0**	**0.629**	**0.629**	0	0	0	0

Table 5. Limit Matrix

4. Solution

The solution is termed the "linking comparison" because it allows the decision maker to impose additional linking connections which will create a strongly connected Supermatrix. The initial pairwise comparisons are completed just as if the matrix were strongly connected; in every instance where an alternative does not possess the criterion the comparison is equal to 0. The subsequent steps are listed below:

1. Raise the weighted Supermatrix to powers.
2. Identify subclusters which are strongly connected.
3. Use the information from the original pairwise comparisons with respect to the influence on the Alternatives from the unweighted Supermatrix to create the "Break out" matrix.
4. Choose a set of criteria to compare.
5. Perform the additional "linking" comparison(s).
6. Renormalize the entries in the "Break out" Supermatrix.

Step 1. Raise the weighted Supermatrix to powers. This step is performed just as it would be done if the Supermatrix were strongly connected.

Step 2. Identify subclusters which are strongly connected. In the resulting limit matrix from Step 1 the weights from the weighted Supermatrix converge to form subclusters (see **Table 5. Limit Matrix**). The term subcluster is used here to refer to a smaller cluster or subnetwork/subgroup of criteria and alternatives which are strongly connected. In Table 3 the Nutella and Banana criteria are strongly connected and hence form a subcluster; the same applies to the Berries and Powdered Sugar, the Milk, the Cocoa and Juice, the Alt 1 and Alt 2, and the Alt 3 and Alt4 subclusters which result in a total of 4 subclusters.

Step 3. Use the information from the original pairwise comparisons with respect to the influence on the Alternatives from the unweighted Supermatrix to create the "Break out" matrix. The eigenvectors in the limit matrix represent the weight, contribution, or influence of the individual criterion with respect to the strongly connected subcluster. From the pairwise comparisons that were already completed the information needed to calculate the individual contribution of each criterion from each alternative can be calculated. The first matrix contains the limiting priorities which were obtained from raising the original weighted Supermatrix to powers. The four columns on the right are the weights from the pairwise comparisons which were performed to measure the influence of the Alternatives on the Toppings and Drinks clusters.

Carbohydrates	Limiting Priorities	Alt 1	Alt 2	Alt 3	Alt 4
Nutella	**0.615**	0.5	0.5	0	0
Banana	**0.217**	0	1	0	0
Berries	**0.129**	0	0	0	1
Powder Sugar	**0.227**	0	0	0.5	0.5
Milk	**0.168**	0.5	0.5	0	0
Cocoa	**0.258**	0	0	1	0
Juice	**0.386**	0	0	0	1

Carbohydrates	Limiting Priorities	Alt 1	Alt 2	Alt 3	Alt 4
Nutella	**0.615**	0.308	0.308	0	0
Banana	**0.217**	0.000	0.217	0	0
Berries	**0.129**	0	0	0	0.129
Powder Sugar	**0.227**	0	0	0.114	0.114
Milk	**0.168**	0.084	0.084	0	0
Cocoa	**0.258**	0	0	0.258	0
Juice	**0.386**	0	0	0	0.386

Table 6. Break out Matrix

The 2nd table in Table 5 is the distribution of the limiting priorities to the alternatives based on the distribution defined from the original pairwise comparisons. While there is a relationship between the priorities within the subclusters there is currently no relationship between the subclusters. This is the relationship that must be imposed by making the pairwise comparisons in Steps 4 and 5.

Step 5. Perform the additional "linking" comparison(s). With the criteria selected the decision maker is ready to make the linking comparison. In a network with n subclusters where $n \geq 2$ the decision maker will need to make at least n-1 linking comparisons. For redundancy purposes the n criteria to be compared can be put into a pairwise comparison matrix where the standard n(n-1)/2 comparisons can be made and checked using the consistency index (Saaty, 1980).

The Banana in Alt 2 will be compared to the Berries in Alt 3. While one may be tempted to simply take the ratio of the two ratios in Table 6 of .217 representing the Banana in Alt 2 and the .129 representing the Berries in Alt 3 (.217/.129) = 1.68 this process would be incorrect because the units in each system are not the same, and therefore the direct comparison is meaningless. By performing the linking comparison the user provides the necessary information to convert the units of one subcluster to that of another subcluster by means of the pairwise comparison process.

This is done by asking the following two questions which are the typical pairwise comparison questions: Which has more Carbohydrates the Banana in Alt 2 or the Berries in Alt 3? How much more Carbohydrates are in the Banana in Alt 2 than the Berries in Alt 3? There are 1.82 times as many Carbohydrates in the Banana than the Berries (see Table 6). The number from this pairwise comparison is then used to renormalize the limiting priorities (see Table 7).

Carbohydrates	Limiting Priorities	Alt 1	Alt 2	Alt 3	Alt 4	**Totals**	**Ratio**
Nutella	0.615	0.308	0.308	0	0		
Banana	0.217	0.000	0.217	0	0	**31**	**1.82**
Berries	0.129	0	0	0	0.129	**17**	
Powder Sugar	0.227	0	0	0.114	0.114		
Milk	0.168	0.084	0.084	0	0		
Cocoa	0.258	0	0	0.258	0		
Juice	0.386	0	0	0	0.386		

Table 7. Linking Comparison

Step 6. Renormalize the entries in the "Break out" Supermatrix. It is worthwhile to note the relationships within the subclusters are identical to the actual relationships among the items within the entire Toppings and Drinks cluster. By setting one of the criterion weights used for the pairwise comparisons in Step 5 equal to 1 and the other to the number representing the ratio from the pairwise comparison, (1.82 in this example), each other priority is simply a ratio of the new entry. In this step it is as though every entry is standardized. It now becomes a ratio of how many of the standardized unit it represents. This is possible because of the definition of the unit of measurement (Choo et al., 1999; Wedley & Choo, 2011). The vectors obtained from Step 6 can be used to interpret the distribution of the criteria among the system

$$C_k / \sum C_k \qquad (2)$$

where C_k represents the total contribution of a single criterion k with respect to the value of the whole system, which is useful information particularly for the sensitivity analysis and the eigenvector of the form

$$a_i / \sum_{i=1}^{n} a_i \qquad (3)$$

which represents the relative contribution of each alternative *a* to the system of n alternatives. The two normalized vectors obtained are equal to the priorities that are obtained in the limit matrix (Table 8).

Carbohydrates	Limiting Priorities	Alt 1	Alt 2	Alt 3	Alt 4	Normalized Vector	Actual Results
Nutella	**0.615**	2.588	2.588	0	0	0.320	0.320
Banana	**0.217**	0.000	1.824	0	0	0.113	0.113
Berries	**0.129**	0	0	0	1.000	0.062	0.062
Powder Sugar	**0.227**	0	0	0.882	0.882	0.109	0.109
Milk	**0.168**	0.706	0.706	0	0	0.087	0.087
Cocoa	**0.258**	0	0	2	0	0.124	0.124
Juice	**0.386**	0	0	0	3	0.185	0.185
Normalized Vector		0.204	0.316	0.178	0.302		
Actual Results		0.204	0.316	0.178	0.302		

Table 8. Normalized Vector compared to Actual Results

This same process should be used in the Fat and Protein subnetworks. The results from the three subnetworks are then combined and synthesized to obtain the final answer. In this model Alt 2 is the preferred alternative with the most energy (Table 9). By performing linking comparisons this weakly connected model was able to be strongly connected and evaluated using the ANP. The final results are equal to the actual results in this tangible model. A special case where the weakly connected alternatives do not share any criteria will be addressed next.

Carbohydrates Limit Matrix

		Toppings				Drinks			Alts			
		Nutella	Banana	Berries	Powder Sugar	Milk	Cocoa	Juice	Alt 1	Alt 2	Alt 3	Alt 4
Toppings	Nutella	0	0	0	0	0	0	0	**0.320**	**0.320**	**0.320**	**0.320**
	Banana	0	0	0	0	0	0	0	**0.113**	**0.113**	**0.113**	**0.113**
	Berries	0	0	0	0	0	0	0	**0.062**	**0.062**	**0.062**	**0.062**
	Powder Sugar	0	0	0	0	0	0	0	**0.109**	**0.109**	**0.109**	**0.109**
Drinks	Milk	0	0	0	0	0	0	0	**0.087**	**0.087**	**0.087**	**0.087**
	Cocoa	0	0	0	0	0	0	0	**0.124**	**0.124**	**0.124**	**0.124**
	Juice	0	0	0	0	0	0	0	**0.185**	**0.185**	**0.185**	**0.185**
Alts	Alt 1	**0.204**	**0.204**	**0.204**	**0.204**	**0.204**	**0.204**	**0.204**	0	0	0	0
	Alt 2	**0.316**	**0.316**	**0.316**	**0.316**	**0.316**	**0.316**	**0.316**	0	0	0	0
	Alt 3	**0.178**	**0.178**	**0.178**	**0.178**	**0.178**	**0.178**	**0.178**	0	0	0	0
	Alt 4	**0.302**	**0.302**	**0.302**	**0.302**	**0.302**	**0.302**	**0.302**	0	0	0	0

Table 9. Limit Matrix

Alternative	Original Data	Linking Comparison Results	Ideal Weight
Alt 1	0.2890	0.2890	0.7990
Alt 2	0.3617	0.3617	1.0000
Alt 3	0.1637	0.1637	0.4526
Alt 4	0.1857	0.1857	0.5133

Table 10. Final Results

As mentioned previously the Drinks cluster is a special form of a disjoint matrix where each alternative only possesses a single criterion. While the criterion Milk is shared by more than one alternative, it is possible to have the case where none of the alternatives share any of the criteria within a cluster of criteria. In the Supermatrix such a cluster could be ordered to appear as an identity matrix. In theory this exception is no different than the general case; rather each criterion is its own subcluster. The criteria from each alternative are then directly compared to each other in Step 4 and the rest of the process is the same. When the new vector is calculated it can be substituted in the new weighted Supermatrix and then used to calculate the limiting priorities.

5. Conclusion

A solution to incorporate the use of disjoint matrices in ANP models has not been addressed previously. This paper provides a relevant example and generalization to demonstrate the advantages that linking comparisons provide the decision maker. First, linking comparisons allow the decision maker to use the current structure of the Supermatrix to analyze decisions with alternatives that are not strongly connected. The second advantage of linking comparisons extends to any ANP model to reduce the cognitive burdens on the decision maker. Cluster weighting and control criteria weighting comparisons can be made at a simpler, more fundamental level within a network at the level of a specific criterion from a single alternative compared to another single criterion from a single alternative. The specific steps are outlined and demonstrated in the example. Five rules are proposed to determine which criteria should be selected to use for the linking comparisons. The results of the Energy Model are presented along with the generalization of the process.

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