A FUZZY AHP MODEL AND ITS APPLICATION TO EVALUATION OF TENDERS

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ABSTRACT

One distinctive characteristic of the Analytical Hierarchy Process (AHP) is that it can test the consistency of pairwise comparison judgments. In this paper, an Objective Consistency Index (OCI) is introduced. By the OCI, pairwise comparison matrix is converted into a particular Linear Goal Programming (LGP) model. In the model, testing consistency, checking and correcting abnormal entries, and finding out relative importance weights of elements are implemented simultaneously. According to the practice of inviting tenders process, a Group decision-making Fuzzy AHP model (GFAHP) is established. The application shows that the model fits the actual decision-making situation friendly.

INTRODUCTION

The AHP (SAATY [3],[4]) is a quantitative method dealing with the problems of evaluation or selection of alternatives under multiple criteria which may be qualitative or quantitative. It has already been successfully applied in a variety of fields in China and the whole world (XU [5]). The problem of evaluating tenders in China is a group decision-making multiple choice problem.

The distinctive characteristic of the AHP is that it can test consistency of people's subjective judgments. If all entries a(i,j) of comparison matrix A are judged and valued accurately, the matrix has three properties. They are

1. All entries on the main diagonal are equal to 1 . i.e.

$$a(i,i)=1 , \quad i,j=1,2,...n \tag{1}$$

2. Reciprocal .

$$a(i,j)=1/a(j,i) , \quad i,j=1,2,...n \tag{2}$$

3. Transitivity .

$$a(i,j)=a(i,k)/a(j,k) , \quad i,j,k=1,2,...,n \tag{3}$$

Actually, these three properties are intrinsic definition of consistency. Properties (1) and (2) are permanently satisfied, but property (3) ,transitivity, is not. There are always some a(i,j) violated equation (3); that is, comparison matrix is almost always inconsistent. Usually, we do not test the consistency of the matrix but the degree of inconsistency. Index

CR based on principal right eigenvalue reflected the macroscopic consistency of comparison matrix. It is subjective and has no actual explanations. There is no clear relationship between CR and the nearness of approximate w to an underlying vector of priorities. When CR is larger than 10%, it means that the degree of consistency is unsatisfied; it does not indicate violated entries and give some correcting approaches. In the actual judgment process, some abnormal entries may not caused by vague judgments but by error entering or imputing. In this case, it is valuable to correct the abnormal entries automatically if CR is near to 10%.

In the process of tender evaluation, if the deviation of each subjectively judged grade of relative importance from an underlying grade is less than one grade (9-point scale used), these subjective judgments are considered satisfyingly consistent. That is, the degree of inconsistency is tolerable.

In this paper, the objective definition of Consistency (OCI) and the LGP model of the comparison matrix (ZHOU [6]) are reviewed systematically. The relationships among OCI,CI and the intrinsic definition are discussed. According to characteristics of tender evaluation, a Group decision-making Fuzzy AHP model (GFAHP) is established. This GFAHP model has been used in a civil construction project.

THE LGP MODEL OF COMPARISON MATRIX

In the AHP, each comparison judgment, whether qualitative, or quantitative, must be quantified according to an appropriate rule. The linear series {k,k=1,2,...,9} is a well known one, but another rule, exponential series {2^(x/2),x=0,1,...,8} is also a simple one. The effectiveness of quantification rule may depend on people's psychological behaviors. From the achievements of experimental psychology, there are two typical patterns of relations between psychological value and physical stimulations, they are FECHNER' logarithm law and STEVENS' power law . In W.S.TOGERSON's opinion, Power (or logarithm) function is generated if people judged according to scale(or difference). In tender evaluation, for every criterion, tenders are not directly compared by their absolute value but by their relative deviations from the promoter's objective and scored according to the deviation scale. The exponential rule may be more similar to actual judgment behavior. So, we prefer to use exponential series as the quantification rule. By the rule, the idea about satisfying consistency can be defined strictly. Assume A=(a(i,j)) is a comparison matrix. a(i,j) is corresponding to a subjective judgment. And a(i,j)=2^(b(i,j)/2). Let r=2^(.5), Then

$$a(i,j)=r^{b(i,j)} \qquad (4)$$

where b(i,j) is the grade of people's judgment (b(i,j)=0,+1,-1,+2 -2,...,+8,-8). Let w=(w(1),...,w(n))' be the wanted underlying vector of priorities. Then, $w(i)/w(j)$ is the estimation of $a(i,j)$. Let $LOG(x)$ be logarithm of x based on r. Then, $LOG(w(i)/w(j))$ is the estimation of grade $b(i,j)$. We have

DEFINITION. (Objective Consistency Index (OCI) . If

$$| LOG(w(i)/w(j))-LOG(a(i,j)) | < 1 \quad \text{for each } i \text{ and } j, \quad (5)$$

Then, consistency of the matrix A is considered as satisfied.

The right-hand side of (5) does not depend on n (the number of elements) or any other parameters. Thus, the definition is objective; and it has practical meaning . The problem of solving priorities becomes an optimal problem as shown below:

Find $w=(w(1),...w(n))'$, so that

$$LOG(w(i)/w(j))-LOG(a(i,j)) \to 0 \;;\; i,j=1,...,n \quad (6)$$

Let

$$w(i)=k*2^{(x(i)/2)} \;;\; i=1,...,n \quad (7)$$

Where constant k makes vector w be normalized.

By the reciprocal axiom, there is no lose of generality when we assume that $i<j$. According to the principle of goal programming (IGNIZIO [2]), problem (6) can be transformed into a LGP model. The LGP model of comparison matrix is

$$\min \sum_{1 \le i < j \le n} (n(i,j)+p(i,j)) \quad (8)$$

$$\text{s.t.} \begin{cases} x(i)-x(j)+n(i,j)-p(i,j)=b(i,j) \\ n(i,j),p(i,j) \ge 0 \end{cases} \quad i,j=1,...,n \quad (9)$$

Where $n(i,j),p(i,j)$ are respectively negative and positive deviation variables. They represent the deviations between the underlying grades and the subjective judgment grades. Suppose $n(i,j),p(i,j),x(i)$ are the optimal solution of model (8) and (9), then

* If $n(i,j),p(i,j)$ are all less than 1, the OCI is satisfied; and the vector w derived from (7) is the wanted vector of priorities.

* If there are several $n(i,j)$ or $p(i,j) \ge 1$, entries correspondingly are considered as abnormal one. In this case, they may be randomly caused by error entering or inputting; and it is valuable to correct these entries automatically, although the OCI is not satisfied. Suppose $n(i,j)$ (or $p(i,j)$) is larger than 1, correspondingly, $b(i,j)$ might be corrected to $b'(i,j)$

$$b'(i,j)=b(i,j)-n(i,j)+p(i,j) . \quad (10)$$

Then, the revised model can be solved as sensitivity analysis in the LP model.

* If there are "many" n(i,j) or p(i,j) >= 1, the OCI is considered too poor to be worth to improve them automatically.

What number means "many" ? Because the rank of a consistency matrix is 1 ,it needs only n-1 entries to generate the whole matrix, thus the total number of corrected entries should be less than n-1, else the matrix would become one established not by people but by a computer program.

There are two theorems about the relations between the OCI and the intrinsic definition of consistency.

THEOREM 1. If A is consistent, then n(i,j),p(i,j) are equal to 0.

PROOF. A is consistent, then, a(i,j)=w(i)/w(j) for all i and j, where w=(w(1),...,w(n))' is the principal right eigenvector. Following from (7), we have

LOG(a(i,j))=x(i)-x(j)

by definition, b(i,j)=LOG(a(i,j)), so, we have

x(i)-x(j)=b(i,j).

that is,

n(i,j)-p(i,j)=0.

By the principle of LGP, n(i,j),p(i,j) are all equal to 0. And x(i),n(i,j),p(i,j) are the optimal solution of (8),(9).

This theorem says that for a consistent comparison matrix, vector derived from LGP model is the same with the principal right eigenvector.

THEOREM 2. If n(i,j),p(i,j) are equal to 0, A is consistent.

PROOF. n(i,j)=0 and p(i,j)=0 for all i and j, then from (9)

x(i)-x(j)=b(i,j) for any i and j.

By (7), we have

LOG(w(i)/w(j))=b(i,j)=LOG(a(i,j)), i.e.

w(i)/w(j)=a(i,j) for any i and j,

then

a(i,j)=w(i)/w(j)=(w(i)/w(k))/(w(j)/w(k)), i.e.

a(i,j)=a(i,k)/a(j,k) for any i, j, and k that is, the transitivity holds.

When the matrix is not consistent, simulations show that if each perturbation of grade b(i,j) is not larger than 1, the CR is always less than 10%(CR=CI/RI, RI of linear series scale is used, actually, it is less than that of exponential series scale).

The LGP model of comparison matrix has 3 characteristics.

1. By the LGP model, testing of consistency, checking and correcting abnormal entries, and finding vector of priorities are carried out simultaneously.

2. Incomplete matrix can also be solved. By the principle of LGP, the model (8) and (9) has always limited optimal solution. In practice, it is convenient to build constraints corresponding to non-negative grades of judgments so that the right-hand side of the model is always non-negative.

3. Constraint coefficients form a sparse matrix. For example, comparison matrix A is

$$\begin{vmatrix} 1 & 1 & 2^{.5} & 2 \\ 1 & 1 & 2^{.5} & 2 \\ 2^{(-.5)} & 2^{(-.5)} & 1 & 2^{.5} \\ 2^{(-1)} & 2^{(-1)} & 2^{(-.5)} & 1 \end{vmatrix},$$

its LGP model is

$$\min \sum_{1 \leq i < j \leq 4} (n(i,j)+p(i,j)) \qquad (11)$$

$$\text{s.t.} \begin{cases} x(1)-x(2) \qquad\qquad +n(1,2)-p(1,2)=0 \\ x(1) \qquad -x(3) \qquad +n(1,3)-p(1,3)=1 \\ x(1) \qquad\qquad -x(4)+n(1,4)-p(1,4)=2 \\ \qquad x(2)-x(3) \qquad +n(2,3)-p(2,3)=1 \\ \qquad x(2) \qquad -x(4)+n(2,4)-p(2,4)=2 \\ \qquad\qquad x(3)-x(4)+n(3,4)-p(3,4)=1 \\ \\ \qquad n(i,j),p(i,j)>=0 \ ,1<=i<j<=4 \end{cases} \qquad (12)$$

Despite of scale of the problem, there are 4 (+1 or -1) nonzero coefficients in each row, and other coefficients are zero.

The LGP model can easily transform fuzzy AHP model into a fuzzy LGP model whose right-hand side includes some fuzzy numbers. It can be converted into a series of conventional LP model corresponding to a series of α-level cut sets.

However, the advantages can not be brought into play, if this particular LGP model is solved by general LP software packages. So, the further research is to find some special algorithm for such sparse LP problem.

A GROUP DECISION-MAKING FUZZY AHP MODEL (GFAHP)

Let 1. G={P1,...,PK} be judgment group, and p=(p(1),...,p(k))' be normalized vector of decision-making influence weights. p(i) can be determined by any appropriate methods including AHP itself;

2. V={2^(x/2),x=0,...,8} be 9-point scale set;

3. C={C1,...,Cn} be the last level of criteria hierarchy. There may be fuzzy information both in individual's subjective judgments and the group itself. Thus,C may have a fuzzy priority vector. It seems that the standard of the best tender is not strict. This is unwelcome currently in the real situation of tender evaluation. Therefore, in this model we did not consider the fuzzy priorities of criteria. We derive hierarchical composed weights of the last level C by conventional approaches;

4. T={T1,...,TS} be the set of alternatives. It is the button level of the whole hierarchy. That is ,it is adjacent to C.

5. FR=(u(i,j)) be a fuzzy rank matrix. Where u(i,j) is the membership degree of Ti rank at jth priority. The fuzzy ranking process is shown as the following.

Let c=(c(1),...,c(n))' be the normalized vector of hierarchical composed priorities;
TCi be comparison matrix of T under criterion Ci; and
TCi[k] be the matrix of TCi established by member k; correspondingly, wCi[k] be its priority vector. Then

$$WCi=(wCi[1],...,wCi[K]),\ i=1,...,n \qquad (13)$$

are SxK matrices of priorities with respect to Ci. The hierarchical composed priority matrix is

$$W=\sum_{1\leqslant i\leqslant n} c(i)*WCi. \qquad (14)$$

W is also a SxK matrix. The column vector w[k] of W is the hierarchical composed priority vector obtained by member k. Each vector corresponds to a point on the super-plate S in space E^n , where super-plate S is

$$w(1)+...+w(n)=1,\ w(i)>=0\ i=1,...,n \qquad (15)$$

Divergence or convergence of the points represents the fuzziness of W. We use FR=(u(i,j)) to describe it, and define

$$u(i,j)=\sum_{1\leqslant k\leqslant K} p(k)*d(i,j)[k] \text{ , for each } i \text{ and } j \qquad (16)$$

where

$$d(i,j)[k]=\begin{cases} 1, & \text{when member } k \text{ rank } Ti \text{ at } j\text{th priority;} \\ 0, & \text{other.} \end{cases} \qquad (17)$$

AN APPLICATION TO TENDER EVALUATION

The project is reconstruction of the Workers' Cultural Palace of Shanghai (ZHOU [6]).

1. It is a state-run project. Members of tender evaluation group G come from different organizations; two of them from General Labor Union of Shanghai, two from the palace, three from general contractor, and one from the design institute of civil construction engineering. As determining weights of criteria, their decision-making weights are equal. But to fuzzy ranking process, they are divided to four sub-group corresponding to four organizations, each sub-group is considered as one person, and its decision-making weight is determined by its number of persons,

   $p=(0.25,\ 0.25,\ 0.375,\ 0.125)'$;

2. Hierarchy of tender evaluation (see Figure 1.)

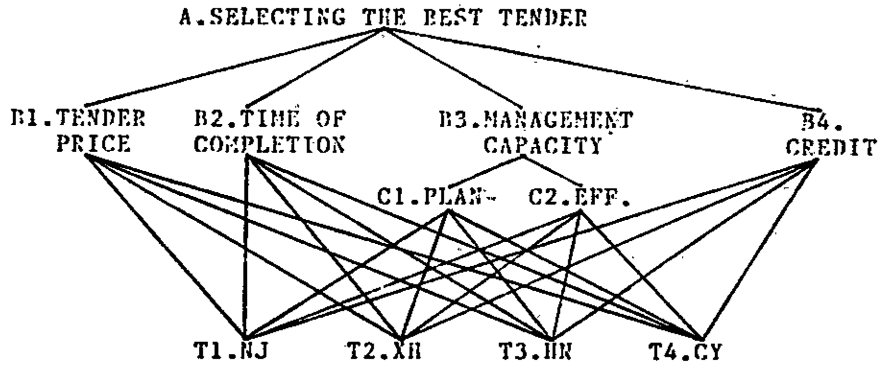

FIGURE 1. HIERARCHY OF TENDER EVALUATION

3. Relative importance weights and hierarchical composed priorities of criteria are

   $A \to B : w=(0.31,0.31,0.22,0.16)'$,
   $B3 \to C : w=(0.667,0.333)'$,
   $A \to C : w=(0.15\ ,0.07\ )'$,
   $A \to \{B1,B2,C1,C2,B4\} : W=(0.31,0.31,0.15,0.07,0.16)'$;

4. Hierarchical composed priority matrix W of tenders is

$$W = \begin{pmatrix} 0.1317 & 0.1230 & 0.1269 & 0.1286 \\ 0.3064 & 0.2897 & 0.3005 & 0.3199 \\ 0.4287 & 0.4529 & 0.4251 & 0.4055 \\ 0.1332 & 0.1343 & 0.1475 & 0.1383 \end{pmatrix} \qquad (18)$$

5. Fuzzy rank matrix FR is

$$
FR = \begin{array}{c|cccc|} & 1st & 2nd & 3rd & 4th \\ T1 & 0 & 0 & 0 & 1 \\ T2 & 0 & 1 & 0 & 0 \\ T3 & 1 & 0 & 0 & 0 \\ T4 & 0 & 0 & 1 & 0 \end{array} \qquad (19)
$$

It means that tender T3 HN is definitely the best one, T2 XH is the second, etc. There are some degree of difference among individual judgments (see (18)); but each membership degree u(i,j) is either 1 or 0. That is, there is no divergence of view about priority of tenders. This result is the same with the actual tender evaluation. The model is well received by general contractor.

CONCLUSION

According to actual decision-making situation, we defined the satisfied consistency index (OCI) that deviations of subjectively judged grades from its underlying grades are all less than one. The definition is objective. By the OCI and exponential series scale, the comparison matrix is converted into a particular LGP model. Fuzzy AHP can also be transformed into a series of the LGP model which corresponds to a series of α-level cut sets of right-hand side. Every such LGP model has the same objective function and sparse constraint coefficient matrix. So the model can be easily solved by a special procedure which utilizes the property of structure. The further research is to improve the algorithm and its software package.

In reality, the process of decision-making is very complex. In the model building, we must consider not only the perfection but also acceptability. The model GFAHP is actually the compromise of both sides. With the development of management capacity the model should be improved further.

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