THE EFFECT OF UNCERTAIN PAIRWISE COMPARATIVE JUDGEMENTS IN THE MULTIPLICATIVE AHP [1]

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Abstract: The multiplicative variant of the analytic hierarchy process (MAHP) employs a method of pairwise comparative judgements by a decision maker to arrive at final impact scores for the alternatives under consideration. This paper examines the effect of imprecision or uncertainty in the decision maker's judgements by expressing each judgement as a probability distribution, and the structure of the MAHP is exploited to derive interval judgements of the alternatives' final impact scores. These interval judgements can be used to determine the probability of rank reversal amongst alternatives, i.e. to assess the stability of the final impact score vector.

Introduction

The multiplicative variant of the analytic hierarchy process (hereinafter referred to as the MAHP) is a multicriteria decision making tool which utilises the concept of pairwise comparisons within a structured hierarchy to arrive at a scoring and rank ordering of the alternatives under consideration (Barzilai *et al*, 1987; Barzilai and Golani, 1991; Barzilai, 1992; Lootsma, 1988, 1993). The decision maker (hereinafter referred to as the DM) provides a subjective cardinal judgement about the intensity of his preference for each alternative over each other alternative under each of a number of criteria or properties. This cardinal judgement is assumed to be a single categorical descriptive judgement taken from a set of such categorical descriptive judgements (and later during the analysis converted on to a numerical scale), implying that the DM can *precisely* express his preferences.

Often, however, a DM might be uncertain about his preference intensity. When the preference judgements contain elements of uncertainty the final impact scores of the alternatives will also be uncertain, and can best be represented by an *impact score interval* (rather than a single point). These impact score intervals might exhibit some degree of overlap, introducing some uncertainty as to the true rank ordering of the alternatives. The family of probabilities of rank reversal in the system offers an indication of the stability of the rank ordering of the alternatives.

Uncertainty on the part of the DM about his preferences in the context of the original (eigenvector-based) AHP was initially studied through a simulation approach (Saaty and Vargas, 1987). A uniformly-distributed variate for each range of the decision maker's preference judgements was generated and used to compute the principal right eigenvector of the pairwise comparison matrices. Repeating this process a large number of times allowed the distribution of the components of the eigenvector to be empirically derived. Thereafter interval estimates for each component were set up, which were used to calculate the probabilities of reversal of ranks within the system of components. Saaty and Vargas' method, whilst tractable, appears cumbersome and impractical: there is a requirement to perform a full-scale simulation each time uncertainty arises in a discrete multicriteria decision problem. Furthermore criticism has been levelled at Saaty and Vargas' approach on several counts (Stam and Silva, 1994). For instance (i) the method used to construct their impact score intervals is questioned (it is dependent on the level of confidence used), and (ii) the impact score interval for each component of the right principal eigenvector is computed independently of that for each other component, ignoring the possibility of correlation between components, which Stam and Silva show may not be insignificant. Stam and Silva propose two further estimators of the probability of rank reversal, and implement them using a simulation approach. They furthermore offer a rigorous statistical analysis of the rank reversal likelihoods in any system.

[1] Freerk Lootsma and Hans Schuijt of the Delft University of Technology offered helpful comments during the preparation of this paper.

Other research to incorporate uncertainty of pairwise judgements in the AHP were restricted to finite interval judgements (Arbel, 1989; Arbel and Vargas, 1993; Zahir, 1991; Salo and Hämäläinen, 1992). Arbel and Vargas propose an optimisation approach, in which the collection of the DM's algebraic preference statements results in a set of linear inequalities, constituting the constraints of a linear programming (LP) problem. Examination of the LP solution provides an indication of the robustness of the rank ordering of the alternatives to the uncertainty in the DM's preference judgements. Arbel and Vargas refer to their method as *preference programming*. Zahir also assumed uniform interval judgements by the DM; his approach was analytical and approximate to the first order for small dimension pairwise comparison matrices (≤ 3 alternatives), and numeric (but exact and computationally more complex) for larger dimensional problems. Zahir does not provide any statistical analysis of the rank reversal problem. Salo and Hämäläinen use preference programming to modify and fine-tune the DM's initially-specified interval judgements in an attempt to minimise inconsistency and ambiguity on the part of the DM, thus attempting to reduce the length of the judgement intervals, and hence increase the stability of the rank ordering. An extension of the AHP using fuzzy logic to aid decision making under vagueness or imprecision has also been proposed (Boender *et al*, 1989; Buckley, 1985; Van Laarhoven and Pedrycz, 1983).

In this paper we will restrict ourselves to the study of uncertainty in the context of the multiplicative variant of the AHP (or MAHP). We will exploit the mathematical structure of the method to determine the theoretical distribution of the alternatives' impact scores when uncertainty is present in the preference judgements amongst the alternatives. Thus interval judgements can be set up for the impact scores, and the probability that alternatives may reverse ranks can be calculated under the given conditions of uncertainty. This method requires only mild distributional assumptions about the uncertainty in the subjective judgements.

In the next section we describe different sources of uncertainty that may arise in a decision making context. Then follows a brief overview of the MAHP, followed by a derivation of the theoretical distribution of the impact scores when uncertainty is present in the preference judgements amongst alternatives. This is followed by a brief discussion of two methods from the literature which measure the stability of the impact scores under uncertainty, and which can be adapted for our purposes.

Sources of Uncertainty in the Decision Making Context

In the decision making context, uncertainty could be categorised into one of three distinct classes:

(1) **Imprecision or Vagueness** The linguistic qualifications describing each category of preference intensity are rather vague or imprecise, so that the DM has difficulty deciding which category best describes his feelings about a comparison. In his mind several of the linguistic qualifications or categories might be more or less appropriate, some more so than others. For example, the DM might know that he prefers Alternative A over Alternative B, but the categories "definite preference" and "strong preference" both seem more or less acceptable descriptions to him.

(2) **Inconsistency** The categories of preference intensity offered to the DM are distinct (i.e. with crisp, non-overlapping endpoints), and are well understood by the DM. However if the DM is asked to provide a number of replications of a specific pairwise comparison under different environmental conditions, his responses are not unique; the range of conditions and his recent experiences create an inability for him to classify his preference intensity into the same single category each time. For example, a DM is asked to offer his preference for the beverages orange juice and coffee. If the session takes place after, say, a convivial dinner, coffee might be preferred. On the other hand if the DM has just partaken in some strenuous exercise on a hot day, orange juice is likely to be preferred.

(3) **Stochastic Judgement** The level of preference intensity depends on some event whose outcome is not known with certainty at the time of the decision. The stochastic nature thus reflects either subjective probabilities that a particular alternative better achieves a given goal, or objective probabilities that reflect uncertain consequences of selecting a particular alternative. For example, a DM is asked to choose one of two investment alternatives that require an identical one-time investment at the beginning of the planning period. The choice might depend on the interest rate ruling over the investment period, which is likely to be unknown at the time of the investment decision.

Although each of the above classes signifies a different form of indecision we will loosely label the above classes under the general heading of "uncertainty". In each case, however, the DM might be reluctant to supply a single value or category to represent the intensity of his preference; the DM's uncertainty can best be described by a *range* of preference judgement values which defines the domain of some probability distribution on the pairwise judgements. This justifies a probabilistic approach which allows the use of standard statistical methodologies to study the effect of uncertainty on the final rank ordering of the alternatives.

The MAHP under Certainty: a Brief Overview

In the basic experiment under **certainty**, two stimuli S_j and S_k (two alternatives A_j and A_k under a particular criterion) are presented to the DM who is requested to express his graded comparative judgement about the pair. That is, he is asked to express his indifference between the two, or his weak, definite, strong or very strong preference for one of them over the other. We assume that stimuli have unknown subjective values V_j and V_k, and the purpose of the experiment is to approximate these values by the calculated impact scores. The DM's pairwise comparative judgement of S_j vis-a-vis S_k is captured on a *category scale* to restrict the range of possible verbal responses, and is characterised by an integer-valued index δ_{jk} designating the gradations of the DM's judgement according to the scale below.

Comparative judgement	Gradation index δ_{jk}
Very strong preference for S_k over S_j	-8
Strong preference for S_k over S_j	-6
Definite preference for S_k over S_j	-4
Weak preference for S_k over S_j	-2
Indifference between S_j and S_k	0
Weak preference for S_j over S_k	+2
Definite preference for S_j over S_k	+4
Strong preference for S_j over S_k	+6
Very strong preference for S_j over S_k	+8

Intermediate integer values can be assigned to δ_{jk} in order to express a hesitation between two adjacent gradations. We will assume that the resulting matrix $\Delta=\{\delta_{jk}\}$ is skew-symmetric, i.e. $\delta_{kj}=-\delta_{jk}$. Thus in an experiment to choose amongst n alternatives, a total of only n(n-1)/2 pairwise judgements are required to fully determine Δ. In the MAHP the DM's judgement about the pair S_j and S_k is used to estimate the *preference ratio* V_j/V_k. The comparative judgements are converted into values on a geometric scale, characterised by scale parameter γ. Thus we define

$$r_{jk} = \exp(\gamma\delta_{jk})$$

to be the numeric estimate of the preference ratio V_j/V_k given by the DM. Although there is no unique scale of human judgement, a plausible value of γ is ln 2, implying a geometric scale with progression factor 2 (Lootsma, 1993).

Suppose that there are n decision alternatives under consideration. We can estimate the vector **V** of subjective stimulus values via logarithmic least squares regression (Lootsma, 1993). We thus approximate **V** by the vector **v** which minimises

$$\sum_{j<k} (\ln r_{jk} - \ln v_j + \ln v_k)^2 \tag{1}$$

Substituting $q_{jk} = \ln r_{jk} = \gamma\delta_{jk}$ and $w_j = \ln v_j$ in (1), the function to be minimised is

$$\sum_{j<k} (q_{jk} - w_j + w_k)^2$$

as a function of the w_j, j=1,....,n. The associated set of normal equations can be written as

$$\gamma \sum_{k=1}^{n} \delta_{jk} = n w_j - \sum_{k=1}^{n} w_k \ , \quad j=1,\ldots,n \tag{2}$$

In principal, by solving (2) we obtain a solution w_j^* with an additive degree of freedom. Thereafter we calculate the vector v^* with components $v_j^* = \exp(w_j^*)$, and use the multiplicative degree of freedom in v^* to find a normalised solution **v**. Note that the rank ordering of the components of **v** does not depend on the scale parameter γ (Lootsma, 1988, 1993). However there is *no unique solution* to the system in (2); by setting $\Sigma w_k=0$ we obtain a particular solution. An unnormalised solution to the logarithmic least squares problem in (1) can then be explicitly written as

$$v_j = \exp(w_j) = \exp\left(\frac{1}{n}\gamma\sum_{k=1}^{n}\delta_{jk}\right) = \prod_{k=1}^{n} r_{jk}^{\frac{1}{n}}$$

i.e. the alternatives' impact scores v_j are found by the calculation of a *geometric mean* of the r_{jk} over all k=1,...,n. The v_j can, if desired, be *normalised* in the sense that $\Sigma v_j=1$. In what follows, however, we will work with the raw (unnormalised) impact scores.

Often the choice of decision alternative has to be taken in the face of multiple conflicting criteria. Under conditions of certainty the alternatives A_j, j=1,...,n are compared pairwise with respect to each criterion C_i, i=1,...,m. Let r_{ijk} represent the numerical value on a geometric scale assigned to the DM's verbal estimate of V_{ij}/V_{ik}, the ratio of the subjective values of the alternatives under consideration under criterion C_i. Then $r_{ijk}= \exp(\gamma\delta_{ijk})$, where δ_{ijk} is the integer-valued index designating the DM's judgement. The impact scores v_{ij} of the alternatives A_j, j=1,...,n under criterion C_i, i=1,...,m are merely the geometric means of the rows of the pairwise comparison matrix under C_i. Final scores for the alternatives in the MAHP follow from the geometric mean aggregation rule: an unnormalised final score s_j for alternative A_j is computed as

$$s_j = \prod_{i=1}^{m} v_{ij}^{g_i} = \prod_{i=1}^{m}\left(\prod_{k=1}^{n} r_{ijk}^{\frac{1}{n}}\right)^{g_i} = \prod_{i=1}^{m}\prod_{k=1}^{n} \rho_{ijk}^{g_i} \tag{3}$$

where $\rho_{ijk} = \exp\left(\frac{\gamma}{n}\delta_{ijk}\right)$, and g_i is the normalised weight assigned to criterion C_i. Sometimes the criterion weights are determined by a pairwise comparison analysis amongst the criteria, analogous to that amongst alternatives under a single criterion. In this case the unnormalised criterion weights are calculated as the geometric means of the row entries in the pairwise comparison matrix of criterion preferences.

Uncertain Pairwise Comparative Judgements in the MAHP

The Distribution of the Components of the Impact Score Vector

Assume now that an individual DM is unable to precisely identify a single value of δ_{ijk} due to some form of uncertainty in his judgement. Under these circumstances it is likely that the DM may prefer to specify a *range* or interval of values for δ_{ijk}. We will assume that the DM's propensity for choosing a value within this range can be modelled by some probability distribution. Thus δ_{ijk} is a random variable, which in turn implies that $r_{ijk} = \exp(\gamma\delta_{ijk})$ is also a random variable. Since

$$\rho_{ijk} = \exp\left(\frac{\gamma}{n}\delta_{ijk}\right),$$

the impact scores s_j in (3) are the product of a *sequence of random variables*. Alternatively we can write

$$\ln s_j = \sum_{i=1}^{m}\sum_{k=1}^{n} g_i \ln \rho_{ijk} = \frac{\gamma}{n}\sum_{i=1}^{m}\sum_{k=1}^{n} g_i\,\delta_{ijk}$$

We will assume that the criterion weights g_i are known or have been set a priori by the decision maker or "problem owner", i.e. there is no uncertainty in the prioritization of the criteria. This assumption is appropriate in a wide range of applications. Under this premise the g_i can be considered constant. Then by a generalisation of the central limit theorem (Feller, 1971), ln s_j is approximately normally distributed. The normal approximation improves as n and m get large and as the distribution of δ_{ijk} tends towards normality. The distribution of the vector of overall impact scores s_j is asymptotically multivariate lognormal under these assumptions. In most practical applications we can expect m and n to be fairly small (both $m,n\leq 10$, say), but the product mn might be fairly large, implying a good approximation of normality. Thus normality of ln s_j will not depend too severely on the distribution of the δ_{ijk}.

Let us assume now that the distribution of δ_{ijk} has mean $\mu_{\delta_{ijk}}$ and variance $\sigma^2_{\delta_{ijk}}$. Then the following results will be approximately true for any symmetrical distribution for δ_{ijk}. Under these circumstances $\ln s_j = \frac{\gamma}{n}\sum_{i=1}^{m}\sum_{k=1}^{n} g_i\,\delta_{ijk}$ will have mean

$$\mu_{\ln s_j} = \frac{\gamma}{n} \sum_{i=1}^{m} \sum_{k=1}^{n} g_i \mu_{ijk}$$

and variance

$$\sigma^2_{\ln s_j} = \left(\frac{\gamma}{n}\right)^2 \left[\sum_{i=1}^{m} \sum_{k=1}^{n} g_i^2 \sigma^2_{\delta_{ijk}} + 2 \sum_{i=1}^{m} \sum_{k=1}^{n} \sum_{p=k+1}^{n} g_i^2 \sigma_{\delta_{ijk}, \delta_{ijp}} \right]$$

where $\sigma_{\delta_{ijk}, \delta_{ijp}}$ is the covariance measured between any pair of elements in the pairwise comparison matrix under criterion C_i. The variance term takes into consideration dependencies within the DM's preference selections themselves. These results can be used to construct a set of interval judgements for the vector of overall impact scores. Thus

$$s_j \in \exp\left[\mu_{\ln s_j} \pm z_{\alpha/2} \cdot \sqrt{\sigma^2_{\ln s_j}}\right] \quad , \quad j=1,2,\ldots,n$$

with 100(1-α)% certainty. Whilst interval judgements of the s_j are of interest in their own right, we seek to compute the probabilities of reversal of rankings amongst the alternatives, which in turn will provide information as to the stability of the impact score vector. This will be based on the degree of overlap within the family of interval judgements for the s_j, following two approaches found in the literature (Saaty and Vargas, 1987; Stam and Silva, 1994), details of which follow in the next section. To this end we can more simply work with the symmetrical interval judgements of ln s_j to examine the stability of the impact scores under uncertainty.

Practical Considerations

In practice the mean and variance of the distributions of each of the δ_{ijk} might be easily elicited by asking the DM to furnish the maximum and minimum δ_{ijk} value that he would be likely to consider. The range R = max(δ_{ijk}) - min(δ_{ijk}) might be taken to represent a 95% coverage of the true range. Then assuming that the distribution of δ_{ijk} is symmetric, $\mu_{\delta_{ijk}}$ may be estimated by the midpoint of R, and $\sigma_{\delta_{ijk}}$ by R/4. Even if the distribution of δ_{ijk} is not symmetric, Chebyshev's Inequality ensures that $\sigma_{\delta_{ijk}}$ is at worst R/9.

Covariance information is much more difficult to elicit. Whilst it is tempting to make the simplifying assumption of perfect independence amongst the δ_{ijk}, this is likely to have the effect of underestimating the variance in the ln s_j. We thus propose the following practical procedure. Consider the entire alternative set under consideration, {A}, and fix all the alternatives except one, say a_p. Offer the DM a range of variants for alternative a_p (call it $a_p^{(r)}$, r=1,2,...,s), and let the DM supply full comparative judgement information for the set of alternatives {A}-a_p+$a_p^{(r)}$, r=1,2,...,s under all criteria C_i, i=1,2,...,m. The resulting response matrices will indicate how the DM's preference selections are correlated, and will allow approximate covariances to be calculated.

Stability of the Vector of Unnormalised Impact Scores under Uncertainty

At least two fundamentally different estimators of the probability of rank reversal of the components of the impact score vector exist, and can easily be adapted for our purposes.

Saaty and Vargas' model (Saaty and Vargas, 1987) advances the construction of a (1-α) level "interval of variation" for the logarithm of the ith component of the impact score vector (IV_i^α), choosing α sufficiently small. If the intersection of IV_i^α and IV_j^α (i≠j) is defined as IV_{ij}^α, then determine the estimate P_{ij} of the probability of rank reversal Π_{ij} associated with each pair of alternatives i and j by

$$P_{ij} = \begin{cases} 0 & \text{if} \quad IV_{ij}^\alpha = \emptyset \\ P(\ln s_j \in IV_i^\alpha) & \text{if} \quad IV_i^\alpha \subseteq IV_j^\alpha \\ P(\ln s_i \in IV_j^\alpha) & \text{if} \quad IV_j^\alpha \subseteq IV_i^\alpha \\ P(\ln s_i, \ln s_j \in IV_{ij}^\alpha) & \text{if} \quad IV_{ij}^\alpha \neq \emptyset \; ; \; IV_i^\alpha, IV_j^\alpha \neq IV_{ij}^\alpha \end{cases}$$

Thus if IV_i^α and IV_j^α do not overlap then i and j will never reverse ranks, and if they do then P_{ij} is the probability that ln s_i and ln s_j are in the intersection IV_{ij}^α.

However IV_i^α depends on the probability level α (if α is made smaller, IV_{ij}^α automatically becomes larger), and P_{ij} implicitly assumes that the components of the impact score vector are statistically independent of one another, which is clearly not the case since the pairwise comparison matrices are known to be skew-symmetric (Stam and Silva, 1994). The extent of dependence will be based on the specific data at hand. Saaty and Vargas' method does, however, allow consideration of the correlations within the DMs' preference responses.

An alternative model by Stam and Silva (Stam and Silva, 1994) is based on the assumption that rank reversal between two alternatives i and j occurs if alternative i would be preferred over j under perfect information, but is calculated to be less preferred based on the sample information on the interval judgements. Then an estimator P_{ij}' of Π_{ij} is found by examining the magnitude of the *difference* between the logarithms of the i[th] and j[th] component of the impact score vector, $D_{ij} = \ln s_i - \ln s_j$. Since the vector of logarithms of the impact scores has been shown to be multivariate normally distributed, D_{ij} is also normal with mean

$$\mu_{D_{ij}} = \mu_{\ln s_i} - \mu_{\ln s_j}$$

and variance

$$\sigma^2_{D_{ij}} = \sigma^2_{\ln s_i} + \sigma^2_{\ln s_j} - 2\,\sigma_{\ln s_i, \ln s_j}$$

where $\sigma_{\ln s_i, \ln s_j}$ is the covariance of the logarithms of the components of the impact score vector. This model thus includes the structural correlations between the components of the impact score vector arising from the skew-symmetry of the pairwise comparison matrices. The estimate of P_{ij}' is given by

$$P_{ij}' = 2\,P(D_{ij} > 0)(1 - P(D_{ij} > 0))$$

The probability $P(D_{ij}>0)$ can be estimated using $\bar{x}_{D_{ij}}$ and $s^2_{D_{ij}}$, the maximum likelihood estimators of $\mu_{D_{ij}}$ and $\sigma^2_{D_{ij}}$ respectively. By this definition P_{ij}' ranges from 0, when one alternative is always preferred to the other, to 0.5, when each alternative is equally likely to be preferred. Whilst Stam and Silva's estimator implicitly includes both sources of correlation in its estimate of rank reversal probability, it may prove virtually impossible in practice to accurately determine the covariance of the logarithms of the components of the impact score vector for a DM.

The probability that alternative j will reverse rank with another alternative is then given by

$$P_j = 1 - \prod_{k \neq j} (1 - P_{jk}) \quad , \quad j = 1, 2, \ldots, n$$

and finally the probability of *at least one* rank reversal occurring in the system is given by

$$P = 1 - \prod_{1 \le j \le k \le n} (1 - P_{jk})$$

The quantity P is a measure of the overall stability of the vector of unnormalised impact scores under uncertainty. However the DM should examine not only P, but also P_j, j=1,...,n and P_{jk}, j,k=1,...,n before reaching a conclusion as to whether or not the level of uncertainty in the system is too great to allow a reliable rank ordering of the alternatives for the purpose at hand. Often the objective of a multicriteria analysis is to identify and rank order just a small number of the top-ranking alternatives (say i out of the n, i<n). All that is important in this case is the probability of rank reversal amongst the top-ranking i alternatives (and possibly the next-ranking one or two): rank reversals amongst the other alternatives are irrelevant.

Conclusions

Uncertainty is encountered in a discrete multicriteria analysis when an individual decision maker is unable to clearly state a preference for one alternative over another. This uncertainty is best represented by an impact score interval (rather than a point estimate) for each alternative under consideration. We have exploited the structure of the MAHP to derive theoretical interval estimates of the geometric row means of the pairwise comparison matrices and of the final impact scores, which allow the stability of the vector of unnormalised impact scores to be assessed under conditions of uncertainty occurring in the preference judgements amongst alternatives. This methodology is a generalisation of the multiplicative AHP under certainty; indeed the MAHP under certainty is merely a special case when all uncertainties equal zero.

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