Dependence of Judgments in Analytic Hierarchy Process: A Statistical Aspect

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Abstract

In Analytic Hierarchy Process, the judgment on a pair of treatments or objects may affect the judgment on other pair of treatments especially when these two pairs have a treatment in common. This dependence could be established by using a model developed by Bahadur [1]. In this model, we test the presence of order effect of presentation within a pair of objects. Any two pairs of alternatives are tested for the correlation between them. It is also tested whether a certain number of pairs have the same degree of correlations. Priority vector, defined in terms of the order effect parameters, is estimated and the hypothesis of equal priority of objects is tested.

Keywords: correlation, dependence, iterative scheme, order effect, priority vector.

1 Introduction

Most of the time judgment on a pair of treatments or objects affect the judgments on other pairs of treatments, especially when these pairs have a treatment in common. Not much work has been done in this area, which recognizes the dependence of the judgments on these pairs of treatments. De Jong [7] estimates the priority vector by the log least squares method, which incorporates this type of dependence. But the model used by him does not accommodate the order effects or correlation aspects.

We use a model developed by Bahadur [1] to establish the dependence of judgments between pairs of treatments. Several authors have previously studied within-pair effect (see [2], [3], [4]), and multivariate comparison experiments (see [5], [6]). In fact, Davidson and Bradley [5] used the model of Bahadur [1] to allow the dependence of judgments among several criteria. There they assumed that the judgments on the pairs of alternatives are independent, while those on several criteria for a fixed pair are dependent. In that paper, an iterative scheme has been given to estimate the priority weights. However, its convergence to the maximum likelihood estimates is not shown. We feel that it is more worthwhile to recognize the dependence of the judgments on the different pairs involving a common alternative.

In section 2, we develop the model involving the parameters of order effect of presentations and correlations between the judgments. Order effects are estimated by the maximum likelihood procedure. We give an iterative scheme for the solution of the likelihood equations. It is also shown that the proposed iterative scheme converges to the maximum likelihood estimates. Section 3 deals with the likelihood ratio test criterion to test whether the order of presentation of the objects to judges plays a significant role. The important issue of correlation of judgments between different pairs is addressed in section 4. In section 5, the priority vector of the objects is estimated and the equality of these weights is also tested in that section.

2 The Model

Let $O_1, O_2, \cdots, O_m$ be m objects which are to be compared pairwise with respect to certain qualitative characteristic by n judges. Consider the vector $\mathbf{X} = (X_{12}, X_{13}, X_{14}, \cdots, X_{m-1,m}, X_{21}, X_{31}, \cdots, X_{m,m-1})'$, where,

$$X_{ij} = \begin{cases} 1 & \text{w.p. } \theta_{ij} \text{ if the first item in } (O_i, O_j) \text{ is preferred,} \\ 0 & \text{w.p. } \phi_{ij} = 1 - \theta_{ij} \text{ if the first item in } (O_i, O_j) \text{ is not preferred.} \end{cases}$$

Set

$$Z_{ij} = \frac{X_{ij} - \theta_{ij}}{\sqrt{\theta_{ij}\phi_{ij}}} \tag{2.1}$$

and define the correlation coefficient between Z_{ij} and Z_{kl}

$$\rho_{ij,kl} = E\left(Z_{ij} Z_{kl}\right), \tag{2.2}$$

Then, the probability density function of $\mathbf{X}$ is given by

$$p(\mathbf{x}) = P(\mathbf{X} = \mathbf{x}) = p_1(\mathbf{x})\, h(\mathbf{x}), \tag{2.3}$$

where,

$$\begin{aligned} p_1(\mathbf{x}) &= \prod_{i \neq j} \theta_{ij}^{x_{ij}}\, \phi_{ij}^{1-x_{ij}}, \\ h(x) &= 1 + \sum_{ij<kl} u(ij,kl)\, \rho_{ij,kl}\, Z_{ij}\, Z_{kl}, \end{aligned}$$

with

$$u(ij,kl) = \begin{cases} 1 & \text{if (i=k or i=l) or (j=k or j=l),} \\ & \\ 0 & \text{otherwise.} \end{cases}$$

Here, $ij < kl$ means that the position of X_{ij} is before the position of X_{kl} in the vector $\mathbf{X}$. In general, the above $p(\mathbf{x})$ need not be a probability density function. Bahadur [1] gives a necessary and sufficient condition for $p(\mathbf{x})$ to be a probability density function. In fact, Bahadur showed that if

$$\lambda_{\min} = 1 - \frac{2}{\Sigma\beta_{ij}}, \text{ where} \beta_{ij} = \max\left\{\frac{\theta_{ij}}{\phi_{ij}}, \frac{\phi_{ij}}{\theta_{ij}}\right\},$$

where λ_{min} is the smallest eigenvalue of the correlation matrix $R = ((\rho_{ij,kl}))$, then it defines a probability density function. We assume that (2.3) is a probability density function. Small values of $\rho_{ij,kl}$s in absolute values would make (2.3) a probability density function. Observe that,

$$Z_{ij} Z_{kl} = \delta(X_{ij}, X_{kl}) \left(\frac{\phi_{ij}}{\theta_{ij}}\right)^{\delta(X_{ij},1)/2} \left(\frac{\phi_{kl}}{\theta_{kl}}\right)^{\delta(X_{kl},1)/2} \tag{2.4}$$

where, $\delta(.,.) = \pm 1$, sign being positive if the arguments agree and negative otherwise.

Denote the elements of the vector $\mathbf{X}$ and corresponding θ and ϕ by Y_i, θ_i and ϕ_i respectively ; $i = 1, 2, \cdots, M$, where, $M = m(m-1)$. In the light of these notations, equations (2.1), (2.2), and 2.4 reduce respectively to

$$
\begin{aligned}
Z_i &= \frac{Y_i - \theta_i}{\sqrt{\theta_i \phi_i}}, \\
\rho_{ij} &= E\left(Z_i Z_j\right), \\
Z_i Z_j &= \delta(Y_i, Y_j) \left(\frac{\phi_i}{\theta_i}\right)^{\delta(Y_i,1)/2} \left(\frac{\phi_j}{\theta_j}\right)^{\delta(Y_j,1)/2}.
\end{aligned}
$$

Then (2.3) can be written in terms of $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_M)'$ as follows.

$$
p(\mathbf{y}) = P(\mathbf{Y} = \mathbf{y}) = p_1(\mathbf{y})\, h(\mathbf{y}), \tag{2.5}
$$

where,

$$
\begin{aligned}
p_1(\mathbf{y}) &= \prod_{i=1}^{M} \theta_i^{y_i} \phi_i^{1-y_i}, \\
h(y) &= 1 + \sum_{i<j} u(i,j)\, \rho_{ij}\, Z_i\, Z_j.
\end{aligned}
$$

Each response consists of a vector of preferences $\mathbf{y} = (y_1, y_2, \cdots, y_M)'$, where component y_i indicates which treatment in the ith pair is preferred, $y_i = 0, 1; i = 1, 2, \cdots, M$. Now let $n(\mathbf{y})$ be the number of times the preference vector $\mathbf{y}$ occurs among the n responses. The logarithm of the likelihood function is then given by

$$
\log L = \sum_{\mathbf{y}} n(\mathbf{y}) \log p(\mathbf{y}),
$$

where $p(y)$ is defined in (2.5) or equivalently,

$$
\log L = \sum_{i=1}^{M} \left[s_i \log \theta_i + (n - s_i) \log(1 - \theta_i)\right] + \sum_{\mathbf{y}} n(\mathbf{y}) \log h(\mathbf{y}), \tag{2.6}
$$

$s_i = \Sigma n(\mathbf{y})\, y_i$, $i = 1, 2, \cdots, M$ and the second sum being over all the possible 2^M values of $\mathbf{y}$ representing the preference responses. Taking derivative of (2.6) with respect to ρ_{ij} ; $i < j$ (with $u(i,j) = 1$), $i, j = 1, 2, \cdots, M$, and with respect to θ_i ; $i = 1, 2, \cdots, M$, we obtain respectively,

$$
\begin{aligned}
\frac{\partial \log L}{\partial \rho_{ij}} &= \sum_{y} \frac{n(\mathbf{y})}{h(\mathbf{y})} \frac{\partial h(\mathbf{y})}{\partial \rho_{ij}}, \\
\frac{\partial \log L}{\partial \theta_i} &= \frac{s_i}{\theta_i} - \frac{n - s_i}{\phi_i} + \sum_{y} \frac{n(\mathbf{y})}{\partial h(\mathbf{y})} \frac{\partial}{\partial \theta_i},
\end{aligned}
$$

where,

$$
\begin{aligned}
\frac{\partial h(\mathbf{y})}{\delta \rho_{ij}} &= u(i,j) \partial(y_i, y_j) \left(\frac{\phi_i}{\theta_i}\right)^{\partial(y_i,1)/2} \left(\frac{\phi_j}{\theta_j}\right)^{\partial(y_j,1)/2} ; i = 1, 2, \cdots, M, \\
\frac{\partial h(\mathbf{y})}{\partial \theta_i} &= -\frac{1}{2\theta_i \phi_i} \left(\frac{\phi_i}{\theta_i}\right)^{\delta(y_i,1)/2} \sum_{j \neq i} u(i,j) \delta(y_j, 1) \rho_{ij} \left(\frac{\phi_j}{\theta_j}\right)^{\delta(y_j,1)/2}.
\end{aligned}
$$

The maximum likelihood estimates $r = \{r_{ij}; i, j = 1, 2, \cdots, M\}$ of the correlation parameters $\rho = \{\rho_{ij}; i, j = 1, 2, \cdots, M\}$ and $q = \{q_i; i = 1, 2, \cdots, M\}$ of the preference probabilities $\theta = \{\theta_i, i = 1, 2, \cdots, M\}$ are obtained by solving the following set of likelihood equations (with $u(i,j) = 1$):

$$\left.\frac{\partial \log L}{\partial \rho_{ij}}\right|_{\rho=r,\theta=q} = 0; i, j = 1, 2, \cdots, M, \tag{2.7}$$

$$\left.\frac{\delta \log L}{\delta \theta_i}\right|_{\rho=r,\theta=q} = 0; i = 1, 2, \cdots, M. \tag{2.8}$$

In the special case of two treatments ($m = 2$), the likelihood equations (2.7) and (2.8) can be solved explicitly to give

$$\begin{aligned} q_i &= \frac{s_i}{n}; i = 1, 2, \\ r &= \frac{n(0,0)n(1,1) - n(0,1)n(1,0)}{\sqrt{s_1 s_2 (n - s_1)(n - s_2)}}. \end{aligned}$$

For $m > 2$, explicit solutions are not possible and equations are solved iteratively for the maximum likelihood estimates. We propose the following iterative scheme to obtain solutions to the set of likelihood equations (2.7) and (2.8). The iterations are indexed by $k, k = 1, 2, \cdots$, since one revised value of each q_i and r_{ij} is obtained for each value of k. Successive values of q will be subindexed by t with only one q_i being revised for each value of t. Details for kth iteration follow in two parts.

(I) A new estimate q of θ is generated cyclically through change of one element of θ at a time. The $(t+1)$st stage value $q^{(t+1)}$ is obtained from the tth stage value through replacement of the element $q_i^{(t)}$ only for which $t = (k-1)M + i - 1, t = (k-1)M, \cdots, t = kM - 1$. Let

$$\begin{aligned} C_i(\mathbf{y}) &= \sum_{j \neq i} u(i,j)\, \delta(y_j, 1)\, r_{ij} \left(\frac{p_j}{q_j}\right)^{\delta(y_j,1)/2}, \\ B_i &= \sum_{y} \frac{n(\mathbf{y})}{h(\mathbf{y})} \frac{\partial h(\mathbf{y})}{\partial q_i} = -\sum_{y} \frac{n(\mathbf{y})\, C_i(\mathbf{y})}{2h(\mathbf{y})\, p_i\, q_i} \left(\frac{p_i}{q_i}\right)^{\delta(y_i,1)/2}. \end{aligned}$$

The iterative equation is defined for the two cases as follows.

Case 1: $B_i < 0$.

$$q_i^{(t+1)} = \left[1 + \frac{p_i}{q_i} \sqrt[4]{\frac{q_i(n - B_i p_i)}{s_i}}\right]^{-1}, \tag{2.9}$$

where, the right hand side values are tth stage values.

Case 2: $B_i > 0$.

$$q_i^{(t+1)} = \left[1 + \frac{p_i}{q_i} \sqrt[4]{\frac{n q_i}{s_i + B_i p_i q_i}}\right]^{-1}, \tag{2.10}$$

where, the right hand side values are t-th stage values.

(II) The system of equations $\left.\frac{\partial \log L}{\partial r_{ij}}\right|_k = 0; i < j, i = 1, 2, \cdots, M$ is solved by the IMSL routine ZXMWD, subject to the condition that the function $h(\mathbf{y})$ be kept positive to give $r^{(k)}$ with each $r_{ij} \in [-1, 1]$. Then $r^{(k)}$ represents the solution to the system of equations for iteration k. Note that $\frac{\partial^2 \log L}{\partial r_{ij}^2}$ is always negative indicating that the likelihood is a convex function of r_{ij}.

Here, $\left.\frac{\partial \log L}{\partial r_{ij}}\right|_k$ denotes the value $\frac{\partial \log L}{\partial r_{ij}}$ obtained when $q^{(km)}$ is substituted for q. The initial estimate $q_i^{(0)}$ is obtained from marginal frequencies $s_i, i = 1, 2, \cdots, M$.

We will now show that the solutions from the iterative schemes (2.9) and (2.10) converge to the maximum likelihood estimates.

Case I. $B_i < 0$.

In this case, we can shown that $\frac{\phi_i^3}{\theta_i^3} B_i$ is an increasing function of θ_i (proved in Appendix- Lemma 2). Now,

$$\begin{aligned} \left.\frac{\partial \log L}{\partial \theta_i}\right|_t &= \left.\frac{\phi_i^4}{\theta_i^4}\left[\frac{\theta_i^3 s_i}{\phi_i^5} - \frac{n\theta_i^4}{\phi_i^5} + \frac{B_i\theta_i^4}{\phi_i^4}\right]\right|_t \\ &= \left.\frac{\phi_i^4}{\theta_i^4}\left[\frac{\theta_i^3 s_i}{\phi_i^5} - \frac{\theta_i^4}{\phi_i^4}\left(\frac{n_i}{\phi_i} - B_i\right)\right]\right|_t \\ &= \frac{\phi_i^4}{\theta_i^4}\left(\frac{n}{\phi_i} - B_i\right)\left[\left(\frac{\theta_i^4}{\phi_i^4}\right)^{(t+1)} - \left(\frac{\theta_i^4}{\phi_i^4}\right)^{(t)}\right], \text{from (2.9)}, \end{aligned}$$

and since $\frac{\theta_i}{\phi_i}$ is an increasing function of θ_i, $\left.\frac{\partial \log L}{\partial \theta_i}\right|_t$ is of same sign as $\Delta\theta_i = \theta_i^{(t+1)} - \theta_i^{(t)}$. Again,

$$\begin{aligned} \left.\frac{\partial \log L}{\partial \theta_i}\right|_{t+1} &= \frac{\phi_i^4}{\theta_i^4}\left(\frac{n}{\phi_i} - B_i\right)\left[\left(\frac{\theta_i^3 s_i/\phi_i^4}{n - B_i\phi_i}\right)^{(t+1)} - \left(\frac{\theta_i^4}{\phi_i^4}\right)^{(t+1)}\right] \\ &= \frac{\phi_i^4}{\theta_i^4}\left(\frac{n}{\phi_i} - B_i\right)\left[\left(\frac{\theta_i^3 s_i/\phi_i^4}{n - B_i\phi_i}\right)^{(t+1)} - \left(\frac{\theta_i^3 s_i/\phi_i^4}{n - B_i\phi_i}\right)^{(t)}\right] \end{aligned}$$

from (2.9), which has the same sign as $\Delta\theta_i$ since

$$\frac{\theta_i^3 s_i/\phi_i^4}{n - B_i\phi_i} = \frac{s_i/\phi_i^2}{(n\phi_i^2 - B_i\phi_i^3)/\theta_i^3}$$

can be seen to be an increasing function of θ_i.

Now, $\frac{\partial \log L}{\partial \theta_i}$ is monotonically decreasing (proved in Appendix-Lemma 1) in θ_i so that $\frac{\partial \log L}{\partial \theta_i}$ has same sign for all values of θ_i between $\theta_i^{(t)}$ and $\theta_i^{(t+1)}$. Thus the change in the likelihood $\Delta \log L = \Delta\theta_i \left.\frac{\partial \log L}{\partial \theta_i}\right|_\epsilon \geq 0$ with equality iff $\Delta\theta_i = 0$, where, $\left.\frac{\partial \log L}{\partial \theta_i}\right|_\epsilon$ denotes $\frac{\partial \log L}{\partial \theta_i}$ at $\theta_i^{(n)} + \epsilon J_i \Delta\theta_i$ for $0 < \epsilon < 1$ and J_i being the vector whose ith element is 1 and all the other elements are equal to 0. Therefore, it is proved that likelihood is increased at every step of the iterative scheme if and only if the corresponding parameter value is changed.

Case II. $B_i > 0$.

In this case, it is shown that $\frac{\theta_i^3 B_i}{\phi_i^3}$ is an increasing function of θ_i (proved in Appendix-Lemma 3). Now,

$$\begin{aligned} \left.\frac{\partial \log L}{\partial \theta_i}\right|_t &= \left.\frac{\phi_i^3}{\theta_i^4}\left[\frac{\theta_i^3 s_i}{\phi_i^4} - \frac{n\theta_i^4}{\phi_i^4} + \frac{B_i\theta_i^4}{\phi_i^3}\right]\right|_t \\ &= \frac{n\phi_i^3}{\theta_i^4}\left[\frac{1}{n}\left(\frac{\theta_i^3 s_i}{\phi_i^4} + \frac{B_i\theta_i^4}{\phi_i^3}\right)^{(t)} - \left(\frac{\theta_i^4}{\phi_i^4}\right)^{(t)}\right] \\ &= \frac{n\phi_i^3}{\theta_i^4}\left[\left(\frac{\theta_i^4}{\phi_i^4}\right)^{(t+1)} - \left(\frac{\theta_i^4}{\phi_i^4}\right)^{(t)}\right], \text{from (2.10)}, \end{aligned}$$

which has the same sign as that of $\Delta\theta_i = \theta_i^{(t+1)} - \theta_i^{(t)}$ since θ_i/ϕ_i is an increasing function of θ_i. Next,

$$
\begin{aligned}
\left.\frac{\partial \log L}{\partial \theta_i}\right|_{t+1} &= \frac{\phi_i^3}{\theta_i^4}\left[\left(\frac{\theta_i^3 s_i}{\phi_i^4} + \frac{B_i\theta_i^4}{\phi_i^3}\right)^{(t+1)} - n\left(\frac{\theta_i^4}{\phi_i^4}\right)^{(t+1)}\right] \\
&= \frac{\phi_i^3}{\theta_i^4}\left[\left(\frac{\theta_i^3 s_i}{\phi_i^4} + \frac{B_i\theta_i^4}{\phi_i^3}\right)^{(t+1)} - \left(\frac{\theta_i^3 s_i}{\phi_i^4} + \frac{B_i\theta_i^4}{\phi_i^3}\right)^{(t)}\right],
\end{aligned}
$$

which has the same sign as $\Delta\theta_i$ since $\frac{\theta_i^3 s_i}{\phi_i^4} + \frac{B_i\theta_i^4}{\phi_i^3}$ is an increasing function of θ_i. Again, since $\theta_i \frac{\partial \log L}{\partial \theta_i}$ is monotone decreasing in θ_i so the result follows as in the case I.

3 Testing Order Effects

It is important to test whether the probability of choosing object O_i in the ordered pair (O_i, O_j) is same as the probability of choosing O_i in the ordered pair (O_j, O_i) for all (O_i, O_j) or not. Thus, the null hypothesis of interest could be formulated as follows.

$$H_0^1 : \theta_{ij} = 1 - \theta_{ji} \ \forall i, j; i \neq j,$$

(or, equivalently $\theta_i = 1 - \theta_{K+i}, i = 1, 2, \cdots, K$, where, $K = m(m-1)/2$).

Under $H_0^1, p(\mathbf{y})$ in (2.5) reduces to

$$p_0(\mathbf{y}) = p_{01}(\mathbf{y}) h_0(\mathbf{y}),$$

where,

$$p_{01}(\mathbf{y}) = \prod_{i=1}^{K} \theta_i^{(1+y_i-y_K+i)} \, \phi_i^{(1-y_i+y_K+i)}.$$

Under H_0^1, the log-likelihood function $L_{01}(\mathbf{y})$ is then given by

$$\sum_{i=1}^{K}(n + s_i - s_{K+i}) \, \log\theta_i + (n - s_i + s_{K+i}) \, \log(1-\theta_i) + \sum_y n(\mathbf{y}) \, \log h_0^1(\mathbf{y}),$$

where,

$$
\begin{aligned}
h_0^1(\mathbf{y}) &= 1 + \sum_{i<j} u(i,j) \, \delta(y_i, y_j) \, \psi_i^{\delta(y_i,1)/2} \, \psi_j^{\delta(y_j,1)/2} \\
\psi_i &= \begin{cases} \phi_i/\theta_i & \text{if } \ i = 1, 2, \cdots, K, \\ \theta_j/\phi_j & \text{if } \ i = K+1, K+2, \cdots, M; j = i - K. \end{cases}
\end{aligned}
$$

The maximum likelihood estimates r_{ij} of $\rho_{ij}; i < j; i, j = 1, 2, \cdots, M$ and q_i of $\theta_i; i = 1, 2, \cdots, K$, under the null hypothesis H_0^1 are obtained by solving the following equations.

$$\sum_y \frac{n(\mathbf{y})}{h_0^1(\mathbf{y})} \frac{\partial h_0^1(\mathbf{y})}{\partial r_{ij}} = 0; i, j = 1, 2, \cdots, M, i < j, \tag{3.1}$$

$$\frac{n + s_i - s_{K+i}}{q_i} - \frac{n - s_i + s_{K+i}}{p_i} + \sum_y \frac{n(\mathbf{y})}{h_0^1(\mathbf{y})} \frac{\partial h_0^1(\mathbf{y})}{\partial q_i} = 0; i = 1, 2, \cdots, K, \tag{3.2}$$

where,

$$\begin{aligned}
\frac{\partial h_0^1(\mathbf{y})}{\partial r_{ij}} &= u(i,j)\delta(y_i,y_j)\psi_i^{\delta(y_i,1)/2}psi_j^{\delta(y_j,1)/2}, \\
\frac{\partial h_0^1(\mathbf{y})}{\partial q_i} &= -\frac{1}{2p_iq_i}\left[\left(\frac{p_i}{q_i}\right)^{\delta(y_i,1)/2}\sum_{j,k\neq i} u(i,j)r_{ij}\delta(y_j,1)\left(\frac{p_j}{q_j}\right)^{\delta(y_i,1)/2}\right. \\
& -r_{i,k+1}\left[\delta(y_i,1)-\delta(y_{k+i},1)\right]\left(\frac{p_i}{q_i}\right)^{(\delta(y_i,1)-\delta(y_{k+i},1))/2} \\
& \left. -\left(\frac{p_i}{q_i}\right)^{\delta(y_{k+i},1)/2}\sum_{j\neq i} u(k+i,j)r_{k+i,j}\delta(y_j,1)\left(\frac{p_j}{q_j}\right)^{\delta(y_j,1)/2}\right].
\end{aligned}$$

Using the similar iterative schemes and similar arguments to show convergence of the iterative schemes to the solutions of (3.1) and (3.2) we get the maximum likelihood estimates of the parameters under the null hypothesis H_0^1.

Let r and q be the maximum likelihood estimates under model and $r^{(01)}$ and $q^{(01)}$ be maximum likelihood estimates of ρ and θ under H_0^1. Then, the likelihood ratio criterion is given by:

$$\lambda_1 = \frac{L_{01}\left(y|r^{(01)}, q^{(01)}\right)}{L\left(y|r,q\right)}. \tag{3.3}$$

The hypothesis H_0^1 is rejected if λ_1 in (3.3) is small or equivalently, $-2\log\lambda_1$ is large. Under H_0^1, $-2\log\lambda_1$ is distributed asymptotically as χ^2 with K df.

4 Correlation of Judgments on Different Pairs

One may be interested in knowing whether the judgment on the pair (O_{i_1}, O_{j_1}) is uncorrelated with the judgment on the pair (O_{k_1}, O_{l_1}). We can test the hypothesis:

$$H_0^2 : \rho_{(i_1,j_1),(k_1,l_1)} = 0 \quad \text{for some } (i_1,j_1),(k_1,l_1).$$

Under the null hypothesis $H_0^2, \log L$ in (2.6) reduces to:

$$\log L_{02} = \sum_{i=1}^{M}\left[s_i\log\theta_i + (n-s_i)\log(1-\theta_i)\right] + \sum_y n(\mathbf{y})\log h_0^2(\mathbf{y}),$$

where,

$$h_0^2(\mathbf{y}) = 1 + \sum_{i<j}{}^{*}\ u(i,j)\rho_{ij}Z_iZ_j,$$

$\sum^*$ denoting the summation over other indices except the particular (i_1,j_1), (k_1,l_1). We get similar sets of likelihood equations. Using the same type of iterative schemes, we find the maximum likelihood estimates of parameters under hypothesis H_0^2.

Let $r^{(02)}$ and $q^{(02)}$ be maximum likelihood estimates of ρ and θ under H_0^2. Then, the likelihood ratio criterion is given by:

$$\lambda_2 = \frac{L_{02}\left(y|r^{(02)}, q^{(02)}\right)}{L\left(y|r,q\right)}. \tag{4.1}$$

The hypothesis H_0^2 is rejected if λ_2 in (4.1) is small or equivalently, $-2\log\lambda_2$ is large. Under $H_0^2, -2\log\lambda_2$ is distributed asymptotically as χ^2 with 1 df.

Sometimes, we may think that the correlations of certain number of pairs of treatments are same. More specifically, we may suspect that correlation between the pair (O_{i_1}, O_{j_1}) and the pair (O_{i_2}, O_{j_2}) is same as the correlation between (O_{i_1}, O_{j_1}) and the pair (O_{i_3}, O_{j_3}) and so on up to the correlation between (O_{i_1}, O_{j_1}) and (O_{i_k}, O_{j_k}). We may test the following null hypothesis:

$$H_0^3 : \rho_{(i_1,j_1),(i_2,j_2)} = \rho_{(i_1,j_1),(i_3,j_3)} = \cdots = \rho_{(i_1,j_1),(i_k,j_k)} = \rho_0, \text{ say.}$$

The $\log L$ in (2.6), under the hypothesis H_0^3, reduces to:

$$\log L_{03} = \sum_{i=1}^{M} [s_i \log\theta_i + (n - s_i)\log(1-\theta_i)] + \sum_y n(\mathbf{y}) \log h_0^3(\mathbf{y}),$$

where,

$$h_0^3(\mathbf{y}) = 1 + \sum_{i<j}{}^{**} u(i,j)\,\rho_{ij}\,Z_i\,Z_j + \rho_0 \sum_{l=1}^{k} u(i_1,j_1)\,Z_{i_l}\,Z_{j_l},$$

$\sum^{**}$ denoting the summation over other indices except the (i_1, j_1), (i_2, j_2); (i_1, j_1), (i_3, j_3); $\cdots$; (i_1, j_1), (i_k, j_k). Then the maximum likelihood estimates of ρ_0 and θ_is respectively are obtained from the following sets of equations:

$$\begin{aligned}
\sum_y \frac{n(\mathbf{y})}{h_0^3(\mathbf{y})} \frac{\partial h_0^3(\mathbf{y})}{\partial \rho_0} &= 0, \\
\frac{s_i - nq_i}{q_i p_i} - \sum_y \frac{n(\mathbf{y})}{h_0^3(\mathbf{y})} \frac{\partial h_0^3(\mathbf{y})}{\partial q_i} &= 0; i = 1, 2, \cdots, M, \\
\frac{1}{2q_i p_i} \left(\frac{q_i}{p_i}\right)^{\delta(y_i,1)/2} u(i,j)\delta(y_j,1) \left(\frac{q_j}{p_j}\right)^{\delta(y_j,1)} &= m_{ij}, \quad p_i = 1 - q_i,
\end{aligned}$$

where,

$$\begin{aligned}
\frac{\partial h_0^3(\mathbf{y})}{\partial \rho_0} &= \sum_{l=1}^{k} u(i_l, j_l) Z_{i_l} Z_{j_l}, \\
\frac{\partial h_0^3(\mathbf{y})}{\partial \theta_i} &= \sum_{j \neq i} m_{ij} r_{ij}^*,
\end{aligned}$$

$$r_{ij}^* = \begin{cases} r_{ij} & \text{if } (i,j) \neq (i_1,j_1),(i_2,j_2);(i_1,j_1),(i_3,j_3);(i_1,j_1),(i_k,j_k), \\ \rho_0 & \text{otherwise.} \end{cases}$$

along with (3.2) for other estimates r_{ij}s of ρ_{ij}s.

Let $r^{(03)}$ and $q^{(03)}$ be maximum likelihood estimates of ρ and θ under H_0^3. Then, the likelihood ratio criterion is given by:

$$\lambda_3 = \frac{L_{(03)}(\mathbf{y}|\mathbf{r}^{(03)}, \mathbf{q}^{(03)})}{L(\mathbf{y}|\mathbf{r}, \mathbf{q})}. \tag{4.2}$$

The hypothesis H_0^3 is rejected if λ_3 in (4.2) is small or equivalently, $-2\log\lambda_3$ is large. Under $H_0^3, -2\log\lambda_3$ is distributed asymptotically as χ^2 with $k-1$ df.

5 Estimation and Test for Equality of Priority Vectors

The estimates of priority vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \cdots, \pi_m)'$ of the alternatives may be found as follows. Define,

$$\mu_i = \sum_{j \neq i} \frac{\theta_{ij} + 1 - \theta_{ji}}{2},$$

$$\pi_i = \frac{\mu_i}{\sum_{j=1}^{m} \mu_j}; i = 1, 2, \cdots, m.$$

The maximum likelihood estimates m_i of μ_i and p_i of π_i are given (with q_{ij}s as estimates of θ_{ij_s}s) respectively by:

$$m_i = \sum_{j \neq i} \frac{q_{ij} + 1 - q_{ji}}{2},$$

$$p_i = \frac{m_i}{\sum_{j=1}^{m} m_j}.$$

The hypothesis of equal priority of the alternatives is:

$$H_0^4 : \pi_1 = \pi_2 = \cdots = \pi_m.$$

Under H_0^4, the number of parameters θ_{ij}s is reduced by $m - 1$. Under H_0^4, estimates q_{ij}s of θ_{ij}s are given by:

$$q_{1i} = q_{i1} + \frac{\sum_{j \neq 1,i}(q_{ij} + 1 - q_{ji}) - \sum_{j \neq 1,i}(q_{1j} + 1 - q_{j1})}{2}; i = 1, 2, \cdots, m.$$

The estimates of the other θ_{ij}s are obtained by (2.7) and (2.8). The hypothesis H_0^4 is rejected if $-2 \log \lambda_4$ is large where, λ_4 is the corresponding likelihood ratio statistic. Under H_0^4, $-2 \log \lambda_4$ is distributed asymptotically as χ^2 with $m - 1$ df.

6 Discussion

In this paper we have generalized the idea of dependence of judgments for Analytic Hierarchy Process.

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Appendix

Lemma 1. $\theta_i \frac{\partial \log L}{\partial \theta_i}$ is monotone decreasing in θ_i except for very small values of θ_i.

Proof. We will show that $\frac{\partial \log L}{\partial \theta_i} + \theta_i \frac{\partial^2 \log L}{\partial \theta_i^2}$ is negative.

$$\frac{\partial \log L}{\partial \theta_i} = \frac{s_i}{\theta_i} - \frac{n - s_i}{1-\theta_i} + \sum_y \frac{n(\mathbf{y})h'(\mathbf{y})}{h(\mathbf{y})},$$

and

$$\frac{\partial^2 \log L}{\partial \theta_i^2} = -\frac{s_i}{\theta_i^2} - \frac{n-s_i}{(1-\theta_i)^2} + \sum_y \frac{n(\mathbf{y})h''(\mathbf{y})}{h(\mathbf{y})} - \sum_y \frac{n(\mathbf{y})\{h'(\mathbf{y})\}^2}{\{h(\mathbf{y})\}^2},$$

where,

$$\begin{aligned} h'(\mathbf{y}) &= -\frac{1}{2\phi_i\theta_i}\left(\frac{\phi_i}{\theta_i}\right)^{\delta(y_i,1)/2} C_i, \\ h''(\mathbf{y}) &= \frac{\delta(y_i,1)+2-4\theta_i}{4\theta_i^2\phi_i^2}\left(\frac{\phi_i}{\theta_i}\right)^{\delta(y_i,1)/2} C_i, \\ C_i &= \sum_{j\neq i} u(i,j)\delta(y_j,1)\rho_{ij}\left(\frac{\phi_j}{\theta_j}\right)^{\delta(y_j,1)/2}. \end{aligned}$$

Then

$$\begin{aligned} \tfrac{\partial \log L}{\partial \theta_i} + \tfrac{\partial^2 \log L}{\partial \theta_i^2} &= \textstyle\sum \frac{n(\mathbf{y})}{h(\mathbf{y})} \frac{C_i}{4\theta_i\phi_i^2} \left(\frac{\phi_i}{\theta_i}\right)^{\delta(y_i,1)/2} \\ & \left[\delta(y_i,1) - 2\theta_i \tfrac{C_i}{h(\mathbf{y})}\left(\tfrac{\phi_i}{\theta_i}\right)^{\delta(y_i,1)/2} - \tfrac{h(\mathbf{y})}{C_i}\left(\tfrac{\theta_i}{\phi_i}\right) 4\theta_i\left(1-\tfrac{s_i}{n}\right)\right], \end{aligned}$$

which is seen to be negative in all the cases. Hence it is proved that

$$\frac{\partial \log L}{\partial \theta_i} + \frac{\partial^2 \log L}{\partial \theta_i^2} < 0.$$

Lemma 2. $\frac{\phi_i^3}{\theta_i^3}\sum_y \frac{n(\mathbf{y})h'(\mathbf{y})}{h(\mathbf{y})}$ is an increasing function of θ_i if B_i is negative.

Proof. $\frac{\phi_i^3}{\theta_i^3}\sum_y \frac{n(\mathbf{y})h'(\mathbf{y})}{h(\mathbf{y})} = \sum n(\mathbf{y})\left[\frac{\phi_i}{\theta_i h(\mathbf{y})}\right]\left[\frac{\phi_i^2 h'(\mathbf{y})}{\theta_i^2}\right].$

$$\frac{\phi_i}{\theta_i h(\mathbf{y})} = \left[\frac{\theta_i}{\phi_i} + \sum_{i<j}\delta(y_i,y_j)\rho_{ij}\left(\frac{\phi_i}{\theta_i}\right)^{\delta(y_j,1)/2}\right]^{-1}$$

$$\frac{\phi_i^2 h'(\mathbf{y})}{\theta_i^2} = \frac{1}{2\phi_i\theta_i} + \left(\frac{\phi_i}{\theta_i}\right)^{2+\delta(y_j,1)/2} C_i.$$

The derivative of $\frac{\phi_i^3}{\theta_i^3}\sum_y \frac{n(\mathbf{y})h'(\mathbf{y})}{h(\mathbf{y})}$ is given by:

$$\sum n(\mathbf{y}) \left[\frac{\phi_i}{\theta_i h(\mathbf{y})}\right] \left[\frac{\delta(y_i,1)/2 + 3 - 2\theta_i}{2\theta_i^2 \phi_i^2}\right] \left(\frac{\phi_i}{\theta_i}\right)^{\delta(y_i,1)/2} C_i$$

$$- \sum n(\mathbf{y}) \, \frac{\phi_i^3}{\theta_i^3} \left[\frac{h'(\mathbf{y})}{h(\mathbf{y})}\right]^2 + \sum n(\mathbf{y}) \, \frac{\phi_i^2}{\theta_i^4} \, \frac{h'(\mathbf{y})}{h(\mathbf{y})}. \tag{6.1}$$

Now, $\delta(y_i, 1)/2 + 3 - 2\theta_i$ is always positive indicating that the first term in the above expression is positive. Combining the second and the third terms we then get,

$$\sum n(\mathbf{y}) \, \frac{\phi_i^2}{\theta_i^4} \, \frac{h'(\mathbf{y})}{h(\mathbf{y})} \left[1 - \theta_i \, \phi_i \, \frac{h'(\mathbf{y})}{h''(\mathbf{y})}\right],$$

which is always positive since $\theta_i \, \phi_i \, \frac{h'(\mathbf{y})}{h(\mathbf{y})} < 1$.

Hence, the sign of (6.1) is always positive. This proves the Lemma 2.

Lemma 3. $\frac{\theta_i^3}{\phi_i^3} \sum_y \frac{n(\mathbf{y}) \, h'(\mathbf{y})}{h(\mathbf{y})}$ is an increasing function of θ_i if B_i is positive.

Proof. $\frac{\theta_i^3}{\phi_i^3} \sum_y \frac{n(\mathbf{y}) h'(\mathbf{y})}{h(\mathbf{y})} = \sum n(\mathbf{y}) \left[\frac{\theta_i}{\phi_i h(\mathbf{y})}\right] \left[\frac{\theta_i^2 h'(\mathbf{y})}{\phi_i^2}\right]$.

$$\frac{\theta_i}{\phi_i h(\mathbf{y})} = \left[\frac{\phi_i}{\theta_i} + \sum_{i<j} \delta(y_i, y_j) \, \rho_{ij} \, \left(\frac{\phi_i}{\theta_i}\right)^{1+\delta(y_j,1)/2}\right]^{-1}$$

$$\frac{\theta_i^2 h'(\mathbf{y})}{\phi_i^2} = \frac{1}{2\phi_i \theta_i} + \left(\frac{\phi_i}{\theta_i}\right)^{\delta(y_j,1)/2-2} C_i.$$

The derivative of $\frac{\theta_i^3}{\phi_i^3} \sum_y \frac{n(\mathbf{y}) h'(\mathbf{y})}{h(\mathbf{y})}$ is given by:

$$\sum n(\mathbf{y}) \left[\frac{\theta_i}{\phi_i h(\mathbf{y})}\right] \left[\frac{\delta(y_i,1)/2 - 1 - 2\theta_i}{2\theta_i^2 \phi_i^2}\right] \left(\frac{\phi_i}{\theta_i}\right)^{\delta(y_i,1)/2} C_i$$

$$- \sum n(\mathbf{y}) \frac{\theta_i^3}{\phi_i^3} \left[\frac{h'(\mathbf{y})}{h(\mathbf{y})}\right]^2 + \sum n(\mathbf{y}) \frac{\theta_i^2}{\phi_i^4} \, \frac{h'(\mathbf{y})}{h(\mathbf{y})}. \tag{6.2}$$

Now, $\delta(y_i, 1)/2 - 1 - 2\theta_i$ is always positive indicating that the first term in the above expression is positive. Combining the second and the third terms we then get,

$$\sum n(\mathbf{y}) \, \frac{\theta_i^2}{\phi_i^4} \, \frac{h'(\mathbf{y})}{h(\mathbf{y})} \left[1 - \theta_i \, \phi_i \, \frac{h'(\mathbf{y})}{h''(\mathbf{y})}\right],$$

which is always positive since $\theta_i \phi_i \frac{h'(\mathbf{y})}{h(\mathbf{y})} < 1$.

Hence, the sign of (6.2) is always positive. This proves the Lemma 3.