# INTERACTIVE DECISION SUPPORT THROUGH INTERVAL JUDGMENTS

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**Abstract** - This paper shows how interval judgments can be integrated into the analytic hierarchy process (AHP) to derive interactive decision support. The interval judgments, which indicate a range of values for the relative importance of factors, allow the decision maker (DM) to enter ambiguous preference statements. Such judgments are synthesized in the hierarchy to obtain weight intervals for the alternatives. As the interval judgments approximate the DM's preferences more and more closely the weight intervals become narrower. The interval AHP thus leads to an iterative process where the DM gradually refines the description of his preferences. The interval AHP is demonstrated by a numerical example, and the INPRE software is outlined to show how the method can be implemented into an interactive decision support tool.

## 1. INTRODUCTION

In the analytic hierarchy process (Saaty, 1977,1980) the DM's preferences are elicited by pairwise comparisons. For each comparison the DM specifies a point estimate which reflects the relative importance of the two factors being compared.

This paper considers the interval AHP, first described in Salo and Hämäläinen (1990a, 1991b), where the DM can enter ranges of numerical values, i.e. intervals, in addition to point estimates when making the pairwise comparisons. The interval judgments permit the DM to make ambiguous statements when he is either unwilling or unable to be explicit about his preferences.

Throughout the process the interval judgments are synthesized in the decision hierarchy to obtain weight intervals for the alternatives. These weight intervals, found by solving a series of linear programming problems, become narrower as the interval judgments describe the DM's preferences more and more precisely. When the weight intervals no longer overlap a complete preference order for the alternatives has been established.

The interval AHP has considerable practical potential. The alternatives' weight intervals can be recomputed at any point of the process so that more interactive decision support can be given to the DM. Moreover, the most preferred alternative may sometimes be found before all the all the interval judgments have been specified. This can substantially reduce the amount of comparison work.

This paper is organized as follows. Section 2 discusses relations between interval judgments and local priorities. Section 3 propagates interval judgments in the hierarchy to obtain weight intervals for the alternatives. Section 4 analyzes earlier statements to help the DM preserve consistency. Section 5 illustrates interval AHP in the context of a car selection problem. It also contains screen displays from the INPRE software, which is an implementation of the interval AHP.

## 2. INTERVAL JUDGMENTS AND LOCAL PRIORITIES

The DM can find it difficult to specify point estimates as required by the AHP. To alleviate such difficulties Saaty and Vargas (1987) propose that a range of values is associated with each pairwise comparison. These statements, called *interval judgments*, capture the subjective uncertainty in the DM's preferences.

The interval judgments allow the DM to incorporate ambiguity into his preference statements. For example, instead of stating that the $i$th subelement is three times as important as the $j$th subelement, the DM can state that the $i$th subelement is at least two but no more than four times as important as the $j$th subelement. This interval judgment is denoted as $I_{ij} = [l_{ij}, u_{ij}] = [2, 4]$.

The interval judgments can be written in matrix form as

$$\begin{pmatrix} 1 & [l_{12}, u_{12}] & \dots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \dots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & \dots & 1 \end{pmatrix} \tag{1}$$

From the reciprocal nature of the pairwise comparisons it follows that $l_{ij}u_{ji} = 1$ for $i \neq j$. The matrix (1) is therefore determined once all the upper (or lower) bounds are known.

Saaty and Vargas (1987) discuss ways of deriving local priorities from (1). They conclude that computing all the right eigenvectors of reciprocal matrices whose elements lie in the intervals $I_{ij}$ is practically an intractable task. The eigenvector is a nonlinear function of the entries of the matrix and no simple method for determining bounds for its components exists. The amount of computation is formidable even if the elements of the comparison matrix are restricted to the first nine integers and their reciprocals.

Yoon (1988) applies the propagation of errors techniques to study the sensitivity of the local priority vector to errors in the comparison matrix. However, he replaces the right principal eigenvector by the normalized row sum of the comparison matrix in order to avoid complicated algebraic calculations.

Arbel (1989) observes that the interval $I_{ij} = [l_{ij}, u_{ij}]$ implies that the $i$th subelement is at least $l_{ij}$ but no more than $u_{ij}$ times as important as the $j$th subelement. Consequently any local priority vector $w = (w_1, \ldots, w_n)$ consistent with this statement must satisfy the constraints $w_i \geq l_{ij} w_j$ and $w_i \leq u_{ij} w_j$. The *feasible region* is the set of local priorities in $Q^n = \{w = (w_1, \ldots, w_n) | w_i \geq 0, \sum_{i=1}^{n} w_i = 1\}$ which satisfy all the constraints resulting from the interval judgments. In his paper Arbel discusses the properties of feasible regions and presents a numerical example.

Arbel and Vargas (1991) suggest a method for deriving local priorities from inconsistent interval matrices and state optimization problems for finding the alternatives' weight intervals. However, for each interval matrix a set of non-linear algebraic equations must be solved to determine the corresponding local priorities. Furthermore, the complexity of the weight interval computation increases rapidly with the size of the hierarchy. Consequently the approach in their paper is numerically infeasible even in small hierarchies.

Salo and Hämäläinen (1990a, 1991b) present the first computationally effective solution to the problem of deriving weight intervals from non-empty feasible regions. The present paper describes this solution, which is based on solving a series of linear programming problems, and shows how the DM's judgments can be examined to help him detect potential inconsistencies.

A related approach to the analysis of value trees is developed in Salo and Hämäläinen (1991c). This method, called PAIRS, employs interval judgments to assess local weights. PAIRS determines a dominance structure for the alternatives by combining ambiguously specified local weights with information about the alternatives' characteristics.

## 3. PROCESSING INTERVAL JUDGMENTS

The feasible region can be written as

$$S = Q^n \cap \{w | w_i \geq l_{ij} w_j, w_i \leq u_{ij} w_j\} \tag{2}$$

where $l_{ij}, u_{ij}$ are the user-specified bounds. As in the AHP these bounds can be restricted to the numbers 1 through 9 and their reciprocals. However, the subsequent results hold even when they are allowed to take values in $(0, \infty)$. The DM may enter these bounds one at a time, and he may also cancel earlier judgments.

This section synthesizes the approximate description of the DM's preferences to derive information about the desirability of the alternatives. The crucial assumption is that at each criterion there is a non-empty feasible region.

Interval arithmetic (see e.g. Moore, 1966) establishes upper and lower bounds for the results of arithmetic operations. However, in the propagation of local priorities interval arithmetic leads to meaningless results. The reason for this is that the components of a local priority vector are not independent as their sum must be equal to one. It is easy to come up with examples where interval arithmetic gives a value greater than one for the upper bound of an alternative's weight.

Instead, we suggest that the upper bound for the weight of an alternative is computed as a solution to a maximization problem, where the objective function is the alternative's weight and the variables are the local priority vectors constrained to the feasible regions. Similarly the lower bound is found by minimizing the alternative's weight subject to the same constraints.

This approach leads to $2n_m$ optimization problems, where $n_m$ is the number of alternatives. Fortunately both the maximization and minimization problems decompose into a series of linear optimization tasks over the feasible regions.

The following notation is introduced to present this decomposition. The number of levels in the hierarchy is $m+1$ with the topmost element on level 0 and the alternatives on level $m$. The number of elements on level $k$ is $n_k$, and $e_{k,l}$ is the $l$th element on level $k$. The set of indexes for the subelements of $e_{k,l}$ is $D_{k,l} \subset \{1, \ldots, n_{k+1}\}$. In other words, $i \in D_{k,l}$ means that $e_{k+1,i}$ is a subelement of $e_{k,l}$.

The feasible region at element $e_{k,l}$ is $S_{k,l}$. Thus any $w \in S_{k,l}$ satisfies the constraints resulting from the interval judgments made among the elements $e_{k+1,i}$, $i \in D_{k,l}$ with respect to $e_{k,l}$. If $w_{k,l} \in S_{k,l}$ and $i \in D_{k,l}$ then $w_{k,l}[i]$ is the share of the (global) weight of $e_{k,l}$ given to element $e_{k+1,i}$. The weight of the element $e_{k,l}$ is denoted by $v_{k,l}$. By convention the weight of the topmost element is one, i.e. $v_{0,1} = 1$.

For a fixed set of local priorities $w_{k,l} \in S_{k,l}$ the weights for the elements are computed as

$$v_{k,l} = \sum_{\{i \in \{1,\ldots,n_{k-1}\} | l \in D_{k-1,i}\}} v_{k-1,i} w_{k-1,i}[l] \tag{3}$$

Thus $v_{k,l}$ does not depend on the priorities $w_{i,j}$ for $i \geq k$. Straightforward calculation shows that $\sum_{i=1}^{n_k} v_{k,i} = 1$.

Let $a \in \{1, \ldots, n_m\}$ so that $e_{m,a}$ is a decision alternative. For any fixed set of local priorities in the feasible regions (3) assigns a weight to $e_{m,a}$. By allowing the local priorities to vary over the feasible regions a set of weights, denoted by $V_a$, is associated with $e_{m,a}$. Theorem 1 gives $V_a$ as a solution to a series of linear programming problems.

**Theorem 1** *At level $m-1$ for $l = 1, \ldots, n_{m-1}$ define*

$$\overline{v}^a_{m-1,l} = \max_{w \in S_{m-1,l}} w[a] \tag{4}$$

$$\underline{v}^a_{m-1,l} = \min_{w \in S_{m-1,l}} w[a] \tag{5}$$

*For $k = m-2,\ldots,0$, $l = 1,\ldots,n_k$ define recursively*

$$\overline{v}^a_{k,l} = \max_{w\in S_{k,l}} \sum_{\{i|i\in D_{k,l}\}} \overline{v}^a_{k+1,i} w[i] \tag{6}$$

$$\underline{v}^a_{k,l} = \min_{w\in S_{k,l}} \sum_{\{i|i\in D_{k,l}\}} \underline{v}^a_{k+1,i} w[i] \tag{7}$$

*Then $V_a = [\underline{v}^a_{0,1}, \overline{v}^a_{0,1}]$.*

*Proof.* Since the feasible regions are closed, bounded sets and the alternative's weight is a continuous function of the local priorities the weight interval $V_a$ is a closed convex subset of $[0,1]$. Let $w_{0,1},\ldots,w_{m-1,n_{m-1}}$ be any feasible local priorities. Then for $k, 0 \le k < m-1$

$$\begin{aligned}
\sum_{i=1}^{n_{k+1}} v_{k+1,i}\overline{v}^a_{k+1,i} &= \sum_{i=1}^{n_{k+1}} \Big(\sum_{\{j|i\in D_{k,j}\}} v_{k,j} w_{k,j}[i]\Big)\overline{v}^a_{k+1,i} \\
&= \sum_{j=1}^{n_k} v_{k,j}\Big(\sum_{\{i|i\in D_{k,j}\}} \overline{v}^a_{k+1,i} w_{k,j}[i]\Big) \\
&\le \sum_{j=1}^{n_k} v_{k,j}\overline{v}^a_{k,j}
\end{aligned}$$

Applying the above inequality repeatedly gives

$$\begin{aligned}
v_{m,a} &= \sum_{i=1}^{n_{m-1}} v_{m-1,i} w_{m-1,i}[a] \\
&\le \sum_{i=1}^{n_{m-1}} v_{m-1,i}\overline{v}^a_{m-1,i} \\
&\le \sum_{i=1}^{n_0} v_{0,i}\overline{v}^a_{0,i} = \overline{v}^a_{0,1}
\end{aligned}$$

Thus $v_{m,a} \le \overline{v}^a_{0,1}$. By the compactness of the feasible regions there exist $\overline{w}_{k,l} \in S_{k,l}$ such that the above inequalities become equalities. The proof for the lower bound is similar. □

Theorem 1 suggests an algorithm for computing the weight intervals for the alternatives. First solve the linear programs (4)-(5) to determine the numbers $\overline{v}^a_{m-1,l}, \underline{v}^a_{m-1,l}$. Then proceed to level $m-2$ and solve problems (6)-(7). Continue to upper levels until the topmost element has been reached.

Modifications to feasible regions at level $k$ do not affect the scalars $\overline{v}^a_{i,j}, \underline{v}^a_{i,j}$ for $i > k$. Thus only levels $0,\ldots,k$ of the hierarchy must be recomputed if the scalars $\overline{v}^a_{i,j}, \underline{v}^a_{i,j}$ are stored for later use. This also means that changes in the upper parts of the hierarchy typically require less computation than changes on the lower levels.

The linear programs (4)-(7) can be solved by enumeration if the set of extreme points of the feasible region $S_{k,l}$ is known. Since $2n_m$ linear programs must be solved at each criterion it may be computationally advantageous to determine the extreme points first. These can be generated by algorithms such as those reported in Matheiss and Rubin (1980). The literature of multicriteria optimization also contains relevant material (see e.g. Steuer, 1986).

The decomposition in theorem 1 holds whenever the feasible regions are closed and convex sets. These requirements are met even if the feasible regions are obtained from a more general description of the DM's preferences. For example, the DM could impose constraints on the components of the local priority vector (e.g. $0.25 \leq w_1 \leq 0.80$) or state that two elements, taken together, are at least as important as a third one (e.g. $w_1 + w_2 \geq w_3$).

However, in the sequel we assume that the constraints on the feasible regions have been derived from interval judgments. This convention is in keeping with the idea of eliciting preferences by pairwise comparisons, which is a fundamental feature of the AHP. Each interval judgment is essentially an ambiguous counterpart to a precisely specified pairwise comparison.

## 4. REFINING INTERVAL JUDGMENTS

It is possible that the DM wants to enter an interval judgment which contradicts the earlier ones. Accepting such a constraint would lead to an inconsistent set of constraints and an empty feasible region so that the approach of section 3 could no longer be applied to synthesize the judgments. To avoid this problem the earlier statements are analyzed to give the DM an a priori characterization of consistent interval judgments.

We make the following technical assumption about the feasible regions

$$\forall\, i \in \{1, \ldots, n\}\ \exists\, w \in S \text{ such that } w_i > 0. \tag{8}$$

The requirement (8) is justified as follows. If (8) did not hold for some $i$ then the weight given to the $i$th subelement would be zero for all the local priority vectors in the feasible region. But then the $i$th subelement has been incorrectly structured because it does not have any impact on the upper level criterion.

For each non-empty feasible region define the intervals $\hat{I}_{ij} = [\hat{l}_{ij}, \hat{u}_{ij}]$ by

$$\hat{u}_{ij} = \max_{w \in S} \frac{w_i}{w_j} \tag{9}$$

$$\hat{l}_{ij} = \frac{1}{\hat{u}_{ji}}. \tag{10}$$

The ratio in (9) is defined to be $\infty$ if $w_i > 0, w_j = 0$ and 0 if $w_i = w_j = 0$. From (8) it follows that there exists a $w \in S$ such that $w_i > 0$, which in turn implies $\hat{u}_{ij} > 0$ so that (10) is well defined. Moreover, if $u_{ij}$ has been specified then $\hat{u}_{ij} \leq u_{ij}$; thus $\hat{I}_{ij} \subseteq I_{ij}$.

Interpreting the results in Potter and Anderson (1980) gives an efficient algorithm for computing the bounds $\hat{u}_{ij}$. Another way to determine these bounds is to employ

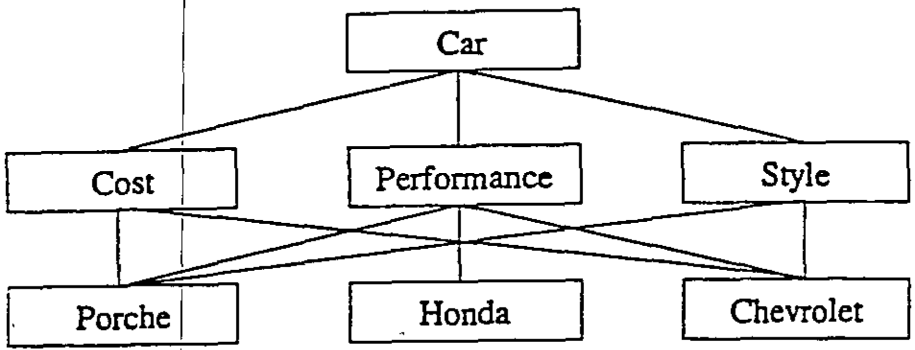


Figure 1: The hierarchy for choosing a car

specialized algorithms for solving linear fractional programming problems or to inspect the extreme points of the feasible region (see e.g. Bazaraa and Shetty, 1979).

Assume that the DM modifies the judgment $I_{ij}$ to $I'_{ij}$ and let $S$ and $S'$ denote the feasible regions before and after the modification. Then, by the results in Potter and Anderson (1980), $S'$ is a non-empty, proper subset of $S$ such that (8) holds for $S'$ if and only if the intervals $I'_{ij}$ and $\hat{I}_{ij}$ overlap. Consequently consistency can be enforced by rejecting judgments which do not intersect $\hat{I}_{ij}$.

The intervals $\hat{I}_{ij}$ also characterize redundant judgments which do not reduce the feasible region. In particular, if $\hat{I}_{ij} \subseteq I'_{ij}$ then the constraints implied by judgments other than $I'_{ij}$ are at least as tight as those corresponding to $I'_{ij}$, i.e. $I'_{ij}$ is redundant. The following theorem summarizes the above results.

**Theorem 2** *$S' \subset S$ and (8) holds for $S'$ if and only if $\emptyset \neq \hat{I}_{ij} \cap I'_{ij} \subset \hat{I}_{ij}$.*

## 5. DECISION SUPPORT BY INPRE

One of the foremost goals in the theoretical development of the AHP is to improve the decision support process. This goal can also be pursued by enhancing the human-computer interaction in decision aids based on the AHP (Hämäläinen and Karjalainen, 1991). Interaction techniques such as graphics and windowing, previously available on expensive workstations only (Salo and Hämäläinen, 1990b, 1991a), can nowadays be implemented on personal computers. To make the most out of these resources we need to focus on computational feasibility and software implementations. We have developed software for the standard AHP (HIPRE v. 2.2 is a fully graphical implementation of the AHP) as well as for feedback modelling (for a description of NETPRE see Hämäläinen and Karjalainen, 1988). INPRE adds the interval AHP to this set of software.

In this section we demonstrate the interval AHP and the related INPRE software in the context of the hierarchy of Figure 1. This hierarchy for a car selection problem is due to Belton and Gear (1984).

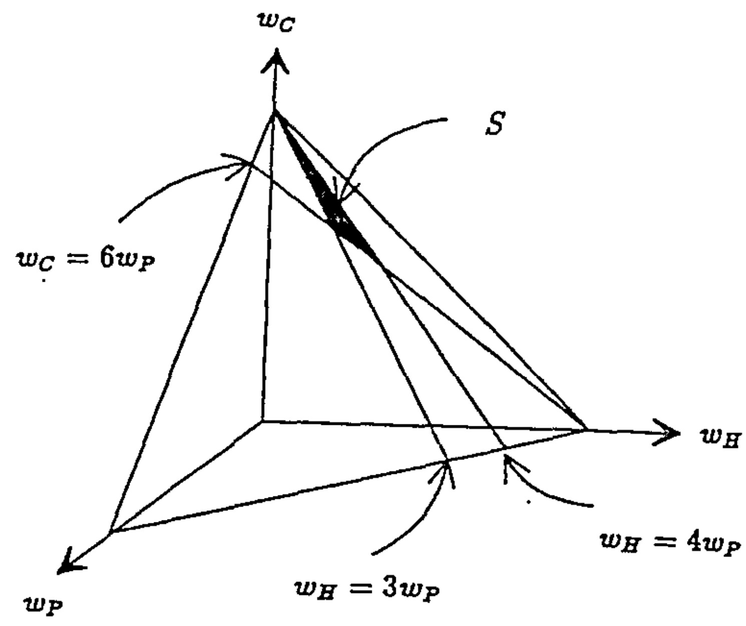


Figure 2: The feasible region at the criterion cost

In the indexes lower case letters denote the criteria. That is, $b$ refers to the topmost element, the overall benefit, and $c, p, s$ refer to the subcriteria cost, performance and style respectively. Upper case letters $P, H, C$ refer to the alternatives Porche, Honda and Chevrolet.

Assume that the DM starts by making the following statements.
With respect to cost,

- Honda is three to four times better than Porche
- Chevrolet is at least six times better than Porche

With respect to performance,

- Porche is at most three times better than Honda
- Porche is at least five times better than Chevrolet
- Honda is at most three times better than Chevrolet

With respect to style,

- Porche is at least three times better than Honda, and five to seven times better than Chevrolet
- Honda is better than Chevrolet

At the cost criterion these interval judgments lead to the constraints $3w_P \leq w_H \leq 4w_P$ and $w_C \geq 6w_P$, which imply $3w_H \leq 2w_C$. Figure 2 shows the feasible region $S_c$

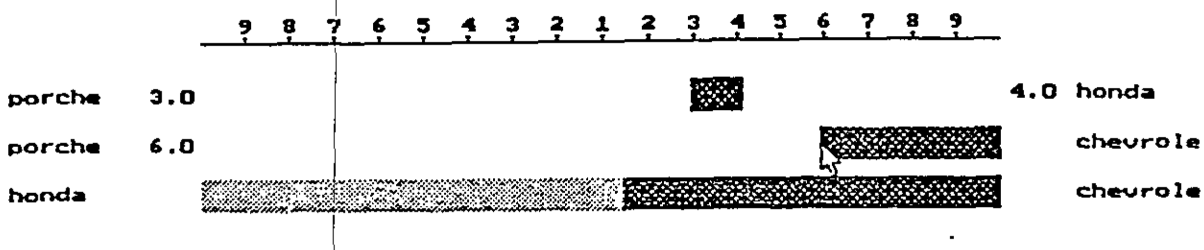


Figure 3: A display for entering interval judgments

corresponding to these constraints. Its extreme points are $(0,0,1), (\frac{1}{10}, \frac{3}{10}, \frac{6}{10}), (\frac{1}{11}, \frac{4}{11}, \frac{6}{11})$. The intervals $\hat{I}_{ij}$, computed from (9)-(10), can be written in matrix form as

$$\hat{I} = \begin{pmatrix} 1 & [\frac{1}{4}, \frac{1}{3}] & [0, \frac{1}{6}] \\ [3, 4] & 1 & [0, \frac{2}{3}] \\ [6, \infty] & [\frac{3}{2}, \infty] & 1 \end{pmatrix} \tag{11}$$

However, the matrix representation in (11) is not very clear. Therefore in the INPRE software another way of displaying interval judgments was adopted. In INPRE the interval judgments are visualized as horizontal bars (see Figure 3). New judgments are entered by moving the ends of these bars with the mouse. The intervals $\hat{I}_{ij}$ are indicated by the darker shaded parts of these bars.

The maximum share of its weight that the criterion cost can give to Honda is $\frac{4}{11}$; thus $\overline{v}_c^H = \frac{4}{11}$ in theorem 1. Examining the feasible regions $S_p$ and $S_s$ gives $\overline{v}_p^H = \frac{1}{3}$ and $\overline{v}_s^H = \frac{7}{31}$. Thus $\overline{v}_b^H = \max_{w \in S_B}(\frac{4}{11}w_c + \frac{1}{3}w_p + \frac{7}{31}w_s)$. At this point no constraining judgments have been made at the topmost criterion so that the maximum weight for Honda is $\overline{v}_b^H = \frac{4}{11} \approx 0.36$. The other bounds for the alternatives' weight intervals are found to be

$$V_P = [0.00; 0.78], \quad V_H = [0.00; 0.36], \quad V_C = [0.08; 1.00].$$

Figure 4, a screen display of INPRE, shows these weight intervals as well as the hierarchy. The numbers at the criteria are values of an ambiguity index (Salo and Hämäläinen, 1991b), which measures the amount of looseness in the characterization of local priorities. The description of the DM's preferences is more precise at those criteria for which the ambiguity index assigns smaller values.

Assume that the DM considers cost to be less important a criterion than performance or style. This statement implies the inequalities $w_c \leq w_p$ and $w_c \leq w_s$, and the extreme points of the feasible region at the topmost criterion are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (0,1,0), (0,0,1)$. Using the numbers $\overline{v}_c^H, \overline{v}_p^H, \overline{v}_s^H$ computed before the maximum weight for Honda is found to be $\frac{1}{3}$. The alternatives' weight intervals become

$$V_P = [0.40; 0.78], \quad V_H = [0.11; 0.33], \quad V_C = [0.08; 0.42].$$

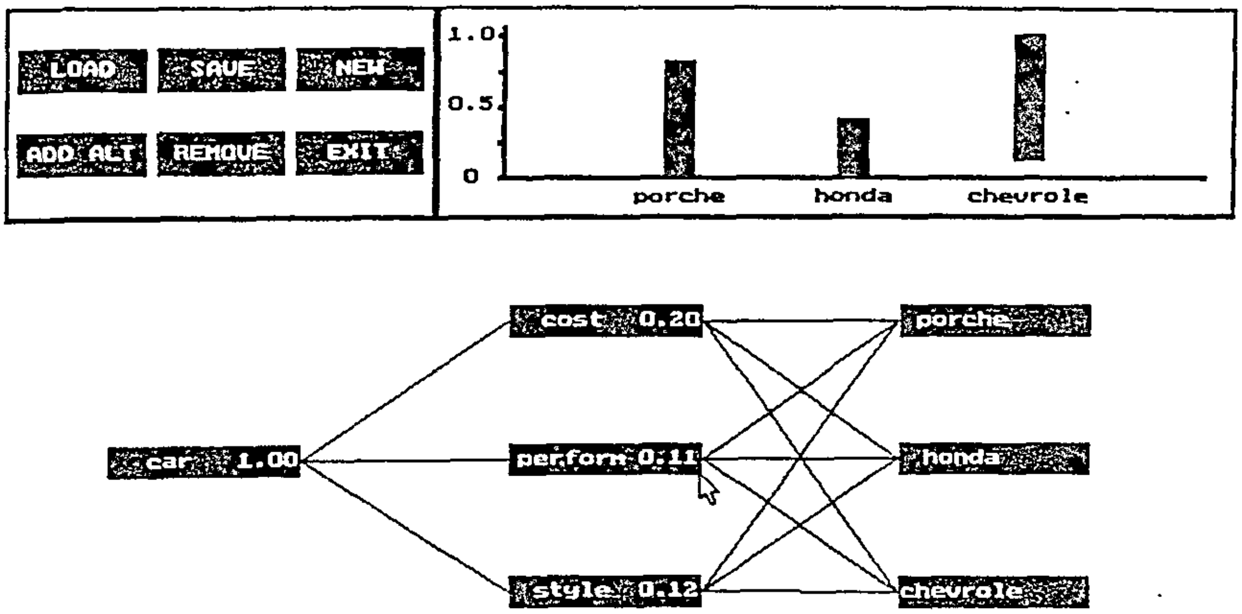


Figure 4: The alternatives' weight intervals in the first phase

Porche now dominates Honda because $V_P$ lies above $V_H$.

Finally, assume that the DM considers performance to be at least three times as important as style. Figure 5 shows how this statement can be entered in INPRE as well as the bounds for the feasible region. After this statement the alternatives' weight intervals, also shown in Figure 5, become

$$V_P = [0.46; 0.71], \quad V_H = [0.15; 0.33], \quad V_C = [0.08; 0.31].$$

At this point the lower bound $\underline{\nu}_3^P$ is greater than the upper bounds for the intervals $V_H$ and $V_C$. This implies that Porche is the most preferred one of the three alternatives.

Note that the most preferred alternative was found without specifying all the bounds of the interval judgments. This indicates that the interval AHP may involve less comparison work than the traditional AHP.

## 6. CONCLUSION

This paper has presented a method for processing interval judgments in the AHP. The interval judgments are synthesized by linear programming to obtain weight intervals for the decision alternatives. Each weight interval consists of weights generated by some set of feasible local priority vectors. During the process, as the DM gradually refines his preferences, these weight intervals become narrower. When they no longer overlap a preference order for the alternatives has been found.

The INPRE software recomputes and visualizes updated weight intervals after each new preference statement. Consequently the interval AHP leads to a more interactive

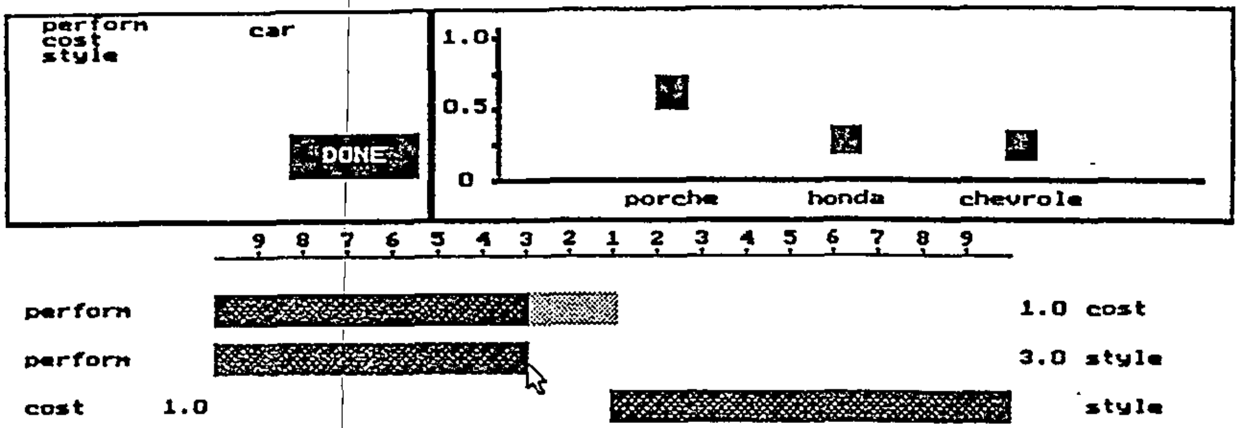


Figure 5: The final weight intervals

decision support process than the traditional AHP. This, in our view, is a significant enhancement to the methodology. Another improvement is that the interval AHP may involve less comparison work because the most preferred alternative may sometimes be identified before eliciting all the interval judgments. Moreover, INPRE shows that the interval AHP can be implemented into operational decision support tools.

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