THE SENSITIVITY ANALYSIS METHOD FOR COMPOSITED PRIORITIES IN THE HIERARCHIC SYSTEMS

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ABSTRACT

In this paper, a kind of the sensitivity analysis method for composited priorities in the hierarchic systems is given.

(I) Introduction

In the hierarchic systems the priorities of elements in any level are derived from pairwise comparison matrices. Obviously, people's preferences, inconsistency of pairwise comparison matrices and errors in calculation may make them lose precision. It is very important to analyze the influence of the errors on the final result which is obtained by the composition priority.

In this paper, the sensitivity analysis method for composited priorities in hierarchic systems is given. We obtain the following results there:

(1) The smaller purterbations of the weight values of the elements in the systems may reversal the rank order of alternatives with respect to focus, we provide the procedure to find the elements which make reverse of rank.

(2) The rule of preservation rank in the composition of priorities is given.

This method is very useful and convenient for decision making in practice.

In Section (II) definition and its calculation method of the limits of rank preservation are given; in Section (III) definition and it's forming method of rank preservation matrices; in Section (IV) sensitivity analizing method for composited priorities by means of the rank preservation matrices is given; and finality in Section (V) we expound character of the method and the problems that need further research.

(II) The limits for rank preservation and sensitive weight vectors

Let n levels in the hierarchic systems be denoted by L_i ($i=1, \cdots, n$), there is a single element in L_1, and there are l_i elements in L_i ($i=2, \cdots, n$). The rank order weights of elements in L_i with respect to the k-th element in L_{i-1} are denoted respectively by

$$w_i^{(i-1,k)} = (w_{i,1}^{(i-1,k)}, w_{i,2}^{(i-1,k)}, \cdots, w_{i,l_i}^{(i-1,k)})^T$$

($k=1, 2, \cdots, l_{i-1}$; $i=n, n-1, \cdots, 2$)

and we construct the following matrices of the impact role of L_i with respect to L_{i-1} by using $w_i^{(i-1,k)}$,

$$W_i^{(i-1)} = \begin{pmatrix} w_{i,1}^{(i-1,1)} & w_{i,1}^{(i-1,2)} & \cdots & w_{i,1}^{(i-1,l_{i-1})} \\ w_{i,2}^{(i-1,1)} & w_{i,2}^{(i-1,2)} & \cdots & w_{i,2}^{(i-1,l_{i-1})} \\ \vdots & \vdots & & \vdots \\ w_{i,l_i}^{(i-1,1)} & w_{i,l_i}^{(i-1,2)} & \cdots & w_{i,l_i}^{(i-1,l_{i-1})} \end{pmatrix}$$

Obviously, the rank order weight vector of alternatives with respect to overall goal is given by,

$$a = [a_1, \cdots, a_{l_n}]^T = W_n^{(n-1)} \cdot W_{n-1}^{(n-2)} \cdots W_2^{(1)}.$$

Let us write simply,

$$A^{(i)} = W_n^{(n-1)} \cdot W_{n-1}^{(n-2)} \cdots W_{i+1}^{(i)}$$

$$= \begin{pmatrix} a_{1,1}^{(i)} & a_{1,2}^{(i)} & \cdots & a_{1,l_i}^{(i)} \\ a_{2,1}^{(i)} & a_{2,2}^{(i)} & \cdots & a_{2,l_i}^{(i)} \\ \vdots & \vdots & & \vdots \\ a_{l_n,1}^{(i)} & a_{l_n,2}^{(i)} & \cdots & a_{l_n,l_i}^{(i)} \end{pmatrix}.$$

$$B^{(i)} = W_{i-1}^{(i-2)} \cdot W_{i-2}^{(i-3)} \cdots W_2^{(1)}$$

$$= (b_1^{(i)}, b_2^{(i)}, \cdots, b_{l_{i-1}}^{(i)})^T$$

($i=n, n-1, \cdots, 1$).

We then have,

$$a = A^{(i)} \cdot W_i^{(i-1)} \cdot B^{(i)}$$

$$a_j = \sum_{s=1}^{l_i} a_{js}^{(i)} \left(\sum_{k=1}^{l_{i-1}} w_{is}^{(i-1,k)} b_k^{(i)} \right) \tag{2.1}$$

($j=1, \cdots, l_n$)

To make discussion be convenient, without loss of generality, we assume that rank order of alternatives with respect to the overall goal is,

$$a(l_n) > a(l_{n-1}) > \cdots > a(2) > a(1)$$

Weights $w_i^{(i-1,k)}$ usually are derived from corresponding pairwise comparison matrices. When some elements in a pairwise comparison matrix are perturbed, it usually leads to that many components of the weight $w_i^{(i-1,k)}$ are changed. We now discuss the influence of these changes on the old rank order of alternatives

Let us suppose, without loss of generality, components $w_{i,s}^{(i-1,k)}$ $(s=1,\cdots,p)$ in $w_i^{(i-1,k)}$ are increasing, and their increments are $\delta_{i,s}^{(i-1,k)}>0$ $(s=1,\cdots,p)$ respectively, but, components $w_{i,t}^{(i-1,k)}$ $(t=p+1,\cdots,l_i)$ are not increasing, and their decrements are $\delta_{i,t}^{(i-1,k)} > 0$ respectively. Because of

$$\sum_{s=1}^{l(i)} w_{i,s}^{(i-1,k)} = 1,$$

obviously is the following,

$$\sum_{s=1}^{p} \delta_{i,s}^{(i-1,k)} = \sum_{t=p+1}^{l_i} \delta_{i,t} \equiv \delta_i^{(i-1,k)}$$

If we denote

$$\bar{a}_j^{(i)} = \max_{1 < s < l_i} \{a_{j,s}^{(i)}\}, \qquad \underline{a}_j^{(i)} = \min_{1 < s < l_i} \{a_{j,s}^{(i)}\},$$

We then have,

$$(\underline{a}_j^{(i)} - \bar{a}_j^{(i)})\,\delta_i^{(i-1,k)} < \sum_{s=1}^{p} a_{j,s}^{(i)}\,\delta_{i,s}^{(i-1,k)}$$

$$- \sum_{t=p+1}^{l_i} a_{j,t}^{(i)}\,\delta_{i,t}^{(i-1,k)} < (\bar{a}_j^{(i)} - \underline{a}_j^{(i)})\,\delta_i^{(i-1,k)},$$

To make the old overall rank order of the alternative with respect to the overall goal not be resersed, the following conditions must be satisfied,

$$a_j + (\bar{a}_j^{(i)} - \underline{a}_j^{(i)})\, b_k^{(i)}\, \delta_i^{(i-1,k)} < a_{j+1} + (\underline{a}_{j+1}^{(i)} - \bar{a}_{j+1}^{(i)})\, b_k^{(i)}\, \delta_i^{(i-1,k)},$$

this is,

$$\delta_i^{(i-1,k)} < \frac{a_{j+1} - a_j}{(\bar{a}_{j+1}^{(i)} - \underline{a}_{j+1}^{(i)}) + (\bar{a}_j^{(i)} - \underline{a}_j^{(i)})\, b_k^{(i)}} \equiv P_i^{(i,k)} \qquad (2.2)$$

We call the P the limit of rank preservation of the weight $w_i^{(i-1,k)}$ for a_j, a_{j+1}. Obviously, when the amount of increments of each component of the weight $w_i^{(i-1,k)}$ is less than the $P_j^{(i,k)}$, the rank order of a_j and a_{j+1} is preserved. When there exist $P_j^{(i,k)} > \delta$ (the error permited), then the corresponding weight vector $w_i^{(i-1,k)}$ is called the sensitive weight vector.

(III) The matrices of rank preservation

Let $i=n, n-1, \cdots, 2$; $k=1, 2, \cdots, l_{i-1}$; $j=n-1, n-2, \cdots, 1$. The limits $P_j^{(i,k)}$ of rank preservation of $w_i^{(i-1,k)}$ can be obtained by formula (2.2), the following matrices can be constructed,

$$P^{(i)} = \begin{pmatrix} P_1^{(i,1)} & P_1^{(i,2)} & \cdots & P_1^{(i,l_{i-1})} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ P_{ln-1}^{(i,1)} & P_{ln-1}^{(i,2)} & \cdots & P_{ln-1}^{(i,l_{i-1})} \\ P^{(i,1)} & P^{(i,2)} & \cdots & P^{(i,l_{i-1})} \end{pmatrix} \quad (i=n, n-1, \cdots, 2) \qquad (3.1)$$

where

$$P^{(i,k)} = \min_{1 \le s \le l_{n-1}} \{ P_s^{(i,k)} \}, \quad (k=1, \cdots, l_{i-1}).$$

We call the (3.1) the matrix of rank preservation with respect to L_{i-1}.

(IV) Sensitivity analysis method

We can make following analyse by using the matrix (3.1):

(1) Analysis in row: Each element in j-th row in the matrix (3.1) gives the limits of rank preservation for a_j, a_{j+1} ($j=1, 2, \cdots, l_{n-1}$). This indicates that the rank order of a_j, a_{j+1} will be preserved, when the amount of increments of each component of the weight vector $w_i^{(i-1,k)}$ ($k=1, \cdots, l_{i-1}$) of elements in L_i is less than the limits $P_j^{(i,k)}$ ($k=1, \cdots, l_{i-1}$) of rank preservation for a_j, a_{j+1} respectively.

(2) Analysis in column: The each element in k-th column in matrix(3.1) gives the limits of rank preservation of $w_i^{(i-1,k)}$ for all a_j, a_{j+1} ($j=1, \cdots, l_{n-1}$). We can see by the matrix (3.1) the influence of the change of $w_i^{(i-1,k)}$ on the old rank for all a_j, a_{j+1} ($j=1, \cdots, l_{n-1}$).

(3) General analysis, the last row in the matrix (3.1) gives the limits of the rank preservation of each weight vector $w_i^{(i-1,k)}$ in L_i for the old overall rank of alternatives with respect to the overall goal When the amount of the increments of all components $w_{i,s}^{(i-1,k)}$ (s=1, ⋯, l_i) of weight vector $w_i^{(i-1,k)}$ is less than the limits $P^{(i,k)}$, the overall rank of all alternatives with respect to the overall goal will be preserved.

(1) If there exist $P^{(i,k)} > \delta$ in the last row in matrix (3.1), then the weight vector $w_i^{(i-1,k)}$ is a sensitive weight vector. It is obviously very important that the accuracy of values of elements in pairwise comparison matrix, adjustment of consistence of that matrix and the accurecy of calculation are carefully examined and determined. Thus the result of decision may be more reliable.

(2) If all $P^{(i,k)} < \delta$ (k=a, ⋯, l_i) are satisfied, then the old rank order of alternatives for the overall goal is stable, that is, reliable.

(V) Conclusion

The sensitivity analysis method given in this paper is simple and useful, and it is easy in software making. This method can also be used in the hierarchic systems with inner dependence. It needs further research how to expand this method to more general circular systems.

REFERENCE

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