UTHER METHOD FOR EXAMINING PAIRWISE COMPARISON MATRIX IN THE AHP

WU WENJIANG

Beijing Graduate School of Wuhan University of Technology

ABSTRACT

For examining consistency of pairwise comparison matrix in the AHP, the paper gives another method, by means of which we can find out values of characteristic polynomial and its derivative without necessarily finding out maximun eigenvalue of matrix.

AHP is useful in processing and analyzing Social—Economic Systems. When we use it, the examination for consistency of pairwise comparison matrix A is needed. So at first we should find out maximum eigenvalue λ_{max} of matrix A, then calculate consistency index $CI=(\lambda_{max}-n)/(n-1)$, and seek out the value of RI from the table, where RI is the average index of randomly generated weights. Finally if the consistency ratio $CR=(CI)/(RI) < 0.1$, it is believed that consistency of matrix A is satisfied. The maximum eigenvalue λ_{max} of matrix A is not accurate if we use general approximate method. This paper gives a rather accurate method for examining consistency of pairwise comparison matrix.

When we consider that $n \times n$ pairwise comparison matrix $A=(a_{ij})$ is positive reciprocal matrix in the following discussion, it means

$$a_{ii}=1; \quad \text{if } i\neq j, \quad a_{ij}=1/a_{ji} > 0, \quad i, j = 1, \ldots, n$$

The characteristic polynomial of matrix A is $f(\lambda) = |\lambda I-A|$, where I is unit matrix. The maximum eigenvalue of matrix A is λ_{max}, and $\lambda_n=n+0.1(n-1)RI$. If $n \geq 3$, then $RI > 0$ and $\lambda_n > n$.

Theorem 1 Consistency of matrix A is satisfied if and only if $\lambda_{max}<\lambda_n$ when $n \geq 3$.

Proof Consistency of matrix A is satisfied, that is

$$CR=(CI)/(RI)=(\lambda_{max}-n)/[(n-1)RI] < 0.1.$$

That is $\lambda_{max}<n+0.1(n-1)RI=\lambda_n$.

Lemma 1 If $\lambda>\lambda_{max}$, $f(\lambda) > 0$ and $f'(\lambda_{max}) > 0$

Proof Since λ_{max} is maximal zero point of $f(\lambda)$, the continuous function $f(\lambda)\neq 0$ when $\lambda > \lambda_{max}$. If $\lambda \to +\infty$, $f(\lambda) \to +\infty$. Therefore if $\lambda>\lambda_{max}$ then $f(\lambda) > 0$. $\therefore f'(\lambda_{max}) \geq 0$. In addition, the maximum eigenvalue λ_{max} of positive matrix must not be repeated root of characteristic polynomial $f(\lambda)$ (See page 170 at reference [1]) $\therefore f'(\lambda_{max})\neq 0$, Overall, $f'(\lambda_{max}) > 0$.

Theorem 2 If $f(\lambda_n) \leq 0$, then consistency of matrix A is not satisfied.

Proof Given $f(\lambda_n) \leq 0$, it follows that $\lambda_n \leq \lambda_{max}$ due to Lemma 1, and according to Theorem 1, it is concluded that consistency of matrix A is not satisfied.

Theorem 3 Given $n\times n$ positive reciprocal matrix A $(n>2)$, if $f(\lambda_n)>0$ and $f^{(k)}(\lambda_n)>0$ $(k=1,2,\ \dots,\ n-3)$, consistency of matrix A is satisfied.

Proof $\because f(\lambda)=|\lambda I-A|=|(\lambda-1)I-(A-I)|=(\lambda-1)^n-[n(n-1)/2](\lambda-1)^{n-2}+\dots$

$\therefore f^{(n-2)}(\lambda)=(n!/2)[(\lambda-1)^2-1],\quad f^{(n-1)}(\lambda)=n!(\lambda-1),\quad f^{(n)}(\lambda)=n!.$

Also since $\lambda_n>n>2$, then $f^{(n-2)}(\lambda_n)$, $f^{(n-1)}(\lambda_n)$ and $f^{(n)}(\lambda_n)$ are all positive. In addition to the given condition of this theorem, $\therefore f(\lambda_n)>0$, $f^{(k)}(\lambda_n)>0$ $(k=1,\ \dots,\ n)$, so that

$$f(\lambda)=f(\lambda_n)+\sum_{k=1}^{n}(1/k!)f^{(k)}(\lambda_n)(\lambda-\lambda_n)^k>0$$

for $\lambda\geq\lambda_n$. We know $f(\lambda_{max})=0$ as well, then $\lambda_{max}<\lambda_n$. Thus it follows that consistency of matrix A is satisfied by Theorem 1.

From Theorem 3 and Theorem 2 we have following theorem.

Theorem 4 Consistency of 3x3 positive reciprocal matrix A is satisfied if and only if $f(\lambda_3)>0$, i.e.

$$(\lambda_3-1)^3-3(\lambda_3-1)>a_{13}/(a_{12}a_{23})+a_{12}a_{23}/a_{13}.$$

Lemma 2 Give $n\times n$ positive reciprocal matrix A, assume $f''(\lambda)>0$ for $\lambda>n$, then consistency of matrix A is satisfied if and only if $f(\lambda_n)>0$ and $f'(\lambda_n)>0$.

Proof $f''(\lambda)>0$ when $\lambda>n$, thus $f'(\lambda)$ is a strictly monotone increasing function. We know $\lambda_{max}\geq n$ (See page 105 at reference[2]), so that $f'(\lambda)>f'(\lambda_{max})$ if $\lambda>\lambda_{max}$. By Lemma 1 we are aware of $f'(\lambda_{max})>0$, consequently, we must have $f'(\lambda)>0$ if $\lambda\geq\lambda_{max}$. It follows that, if $f'(\lambda_n)\leq 0$, $\lambda_n<\lambda_{max}$ certainly. By virtue of Theorem 1, consistency of matrix A is not satisfied. If $f(\lambda_n)\leq 0$, from Theorem 2 also consistency of matrix A is not satisfied. Thus the necessity of condition in the theorem has been established.

Now the sufficiency of condition in the theorem will be proved. Since $f'(\lambda)$ is a strictly monotone increasing function for $\lambda>n$, $f'(\lambda)$ has one zero point at most for $\lambda>n$.

1) Let $f'(\lambda)$ has not zero point when $\lambda>n$. And by Lemma 1 we know $f'(\lambda_{max})>0$, where $\lambda_{max}\geq n$, it follows that $f'(\lambda)>0$ when $\lambda>n$ from continuity of $f'(\lambda)$, this implies that $f(\lambda)$ is a strictly monotone increasing function for $\lambda>n$. Consequently, if $n<\lambda<\lambda_{max}$, $f(\lambda)<f(\lambda_{max})=0$.

2) $f'(\lambda)$ has only one zero point λ_0 when $\lambda>n$. And by virtue of $f'(\lambda)>0$ when $\lambda\geq\lambda_{max}$, we have $n<\lambda_0<\lambda_{max}$. Since $f'(\lambda)$ is a strictly monotone increasing function for $\lambda>n$, so if $n<\lambda<\lambda_0$, $f'(\lambda)<f'(\lambda_0)=0$; if $\lambda_0<\lambda<\lambda_{max}$, $f'(\lambda)>f'(\lambda_0)=0$, that is, $f(\lambda)$ is a strictly monotone increasing function, thus $f(\lambda)<f(\lambda_{max})=0$.

By 1) and 2), it is impossible that both $f(\lambda)$ and $f'(\lambda)$ are positive for $n<\lambda<\lambda_{max}$. Therefore, if $f(\lambda_n)>0$ and $f'(\lambda_n)>0$ it will be

$\lambda_n > \lambda_{max}$ certainly. It follows that consistency of matrix A is satisfied by Theorem 1.

Theorem 5 Consistency of 4x4 positive reciprocal matrix A is satisfied if and only if $f(\lambda_4) > 0$ and $f'(\lambda_4) > 0$.

Proof $f(\lambda)=|\lambda I-A|=|(\lambda-1)I-(A-I)|=(\lambda-1)^4-6(\lambda-1)^2-a(\lambda-1)+c,$

where $c=|A-I|,$

$$a=\sum_{i}\sum_{j}\sum_{k}\left(x_{ijk}+\frac{1}{x_{ijk}}\right),\quad x_{ijk}=a_{ij}a_{jk}/a_{ik},\qquad i,j,k=1,2,3,4$$

$\therefore f''(\lambda)=12[(\lambda-1)^2-1]$, thus if $\lambda>4$, $f''(\lambda)>0$.

Through Lemma 2 we have Theorem 5 immediately.

Theorem 6 Given 5x5 positive reciprocal matrix, assume $f''(5)>0$, then consistency of matrix A is satisfied if and only if $f(\lambda_5)>0$ and $f'(\lambda_5)>0$.

Proof $f(\lambda)=|(\lambda-1)I-(A-I)|=(\lambda-1)^5-10(\lambda-1)^3-d_1(\lambda-1)^2+b_1(\lambda-1)-c_1$

where $c_1=|A-I|$

b_1 is the sum of five 4x4 principal minors of matrix $(A-I)$

$$d_1=\sum_{i<j<k}(x_{ijk}+1/x_{ijk}),\quad x_{ijk}=a_{ij}a_{jk}/a_{ik}\qquad i,j,k,=1,2,3,4,5$$

It is obvious that $f'''(\lambda)=60[(\lambda-1)^2-1]$, which implies that $f''(\lambda)$ is a strictly monotone increasing function, in addition to given $f''(5)>0$, we have that if $\lambda>5$ then $f''(\lambda)>f''(5)>0$. With Lemma 2 we can conclude that Theorem 6 holds.

REFERENCE

[1] Thomas L. Saaty, The Analytic Hierarchy Process, Mc Graw — Hill, Inc., 1980.

[2] Zhao Huan-chen, Xu Shu-buo, He Jing-shen, The Analytic Hierarchy Process, Science Publishing House, 1986.