ELICITING THE RELATIVE WEIGHTS FROM INCOMPLETE RECIPROCAL MATRICES

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ABSTRACT

In the AHP, a reciprocal matrix is constructed from all pairwise comparison judgments of the DM. Some elements of the reciprocal matrix may be obtained with greater confidence than others if the DM is forced to answer all pairwise comparisons. The present paper proposes a procedure for eliciting the relative weights using "incomplete" reciprocal matrix. It does not require the DM to answer all the pairwise comparisons. The DM needs to answer only those pairs which he/she feels comfortable or confidenct to do so.

1. Introduction

The eigenvector method proposed by Saaty(see, for instance, Saaty[2] and Saaty & Vargas[3]) has provided a simple, but mathematically elegant, means to assess the relative weights from a reciprocal matrix. Since its introduction, many applications have been reported in various fields(for instance, see Zahedi[6]).

All pairwise comparison judgments of the DM are required to construct the reciprocal matrix. In the AHP, all elements of the reciprocal matrix are assumed to have the same confidence. As the judgments in pairwise comparisons are dependent, to a large extent, on personal experience, learning, situations and state of mind, the degree of easiness or confidence to make the judgments on the ratios can be different. Therefore, some elements of the reciprocal matrix may be obtained with greater confidence than others if the DM is forced to answer all pairwise comparisons. Furthermore, when the number of pairwise comparisons is large, it could be a great burden to require the DM to express all pairwise comparisons.

The present paper proposes a procedure for eliciting the relative weights using "incomplete" reciprocal matrix. Our procedure allows the DM to skip some difficult pairwise comparisons. It must be, however, stressed that it does not mean that the number of pairwise comparisons should be as least as possible. As will be seen, when the confidence is almost the same, the "complete" reciprocal matrix is most desirable.

2. Preliminaries

To begin with, let us give an example to show how to elicit an eigen weight vector from an incomplete reciprocal matrix.

Example 1. Suppose that there are five criteria and that we have the following revealed weight ratios.

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & & 3 & 5 & 7 & 9 \\ 2 & & & & 4 & 5 \\ 3 & & & & 4 & \\ 4 & & & & & 3 \\ 5 & & & & & \end{array} \qquad (1)$$

By using the reciprocal property, we have

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & & 3 & 5 & 7 & 9 \\ 2 & 1/3 & & & 4 & 5 \\ 3 & 1/5 & & & 4 & \\ 4 & 1/7 & 1/4 & 1/4 & & 3 \\ 5 & 1/9 & 1/5 & & 1/3 & \end{array} \qquad (2)$$

Putting $a_{ii}=1$, $i=1,2,\dots,5$, and replacing the missing values of a_{ij} by w_i/w_j respectively, we have the following equations for solving the corresponding eigenvalue and eigenvector.

$$\begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 1/3 & 1 & w_2/w_3 & 4 & 5 \\ 1/5 & w_3/w_2 & 1 & 4 & w_3/w_5 \\ 1/7 & 1/4 & 1/4 & 1 & 3 \\ 1/9 & 1/5 & w_5/w_3 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} \qquad (3)$$

or

$$\begin{aligned} w_1 + 3w_2 + 5w_3 + 7w_4 + 9w_5 &= \lambda w_1 \\ 1/3w_1 + 2w_2 + 4w_4 + 5w_5 &= \lambda w_2 \\ 1/5w_1 + 3w_3 + 4w_4 &= \lambda w_3 \\ 1/7w_1 + 1/4w_2 + 1/4w_3 + w_4 + 3w_5 &= \lambda w_4 \\ 1/9w_1 + 1/5w_2 + 1/3w_4 + 2w_5 &= \lambda w_5 \end{aligned} \qquad (4)$$

Rewriting (4) in the matrix form, we have

$$\begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 1/3 & 2 & 0 & 4 & 5 \\ 1/5 & 0 & 3 & 4 & 0 \\ 1/7 & 1/4 & 1/4 & 0 & 3 \\ 1/9 & 1/5 & 0 & 1/3 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} \qquad (5)$$

Note that the coefficient matrix of (5) is obtained by putting 0's in the non-diagonal blank positions of (2) and the number of the row blank positions of (2) in each diagonal elememt which is equal to 1 plus the number of 0 in the corresponding row. Later we formally define the coefficient matrix so defined as the *derived reciprocal matrix* of the original matrix.

Solving the eigenvalue problem of (5), we obtain

$$\tilde{\lambda}_{max} = 5.209$$

$$\tilde{W} = (.531\ .197\ .168\ .067\ .038)$$

where $\tilde{\lambda}_{max}$ is the maximum eigenvalue of (5), and $\tilde{W}$ is the corresponding eigenvector.

3. Eigen Weight Method based on Incomplete Reciprocal Matrices

Let $Q = \{1, 2, \ldots, q\}$ be the index set of q criteria. Given a set of the revealed weight ratios $\{a_{ij}\}$, let $\mathcal{I} = \{I_1, I_2, \ldots, I_n\}$ be the collection of the pair indices such that $i \neq j$ and if $I = (i, j) \in \mathcal{I}$ then a_{ij} is a revealed weight ratio. (Thus $a_{ij} > 0$) As we are interested in reciprocation, given $\mathcal{I}$, we denote its *reciprocally expanded set* by $\hat{\mathcal{I}}$ That is,

$$\hat{\mathcal{I}} = \{(i, j) \mid (i, j) \in \mathcal{I}, \text{or } (j, i) \in \mathcal{I}\}$$

Definition 1. (i)Let $A=(a_1, a_2)$ and $B=(b_1, b_2)$ be two elememts of $\mathcal{I}$. We say A and B are *overlapping* iff $A \neq B$ and $\{a_1, a_2\} \cap \{b_1, b_2\} \neq \phi$. (ii) $\mathcal{I}$ is said to be *connected* iff for any A and B of $\mathcal{I}$ there is a sequence $I_1, I_2, \ldots, I_s$ of $\mathcal{I}$ such that I_{l-1} overlaps $I_l, l=2,\ldots,s$, and $I_1=A$ and $I_s=B$.

Recall that, given a pair index set $\mathcal{I}$, we denote its reciprocally expaded set by $\hat{\mathcal{I}}$. Note that $\mathcal{I}$ is connected iff $\hat{\mathcal{I}}$ is connected. Furthermore, if $\mathcal{I}$ is connected, then the sequence in the Def. 1, (ii) can be replaced by a directed path in $\mathcal{I}$. (Thus, from any state i of Q we can find a directed path $\{(i, i_1), (i_1, i_2), \ldots, (i_t, k)\}$ to go any other state k of Q).

Definition 2. $\mathcal{I}$ is a *covering* of Q if (i) $\mathcal{I}$ is connected, (ii) Q is contained by the union of the elements of $\mathcal{I}$.

Example 2. Let $Q=\{1, 2, 3, 4, 5\}$. Then $\mathcal{I}_1=\{(1, 2), (1, 3), (1, 4), (1, 5)\}$, $\mathcal{I}_2=\{(1, 2), (3, 2), (4, 3), (4, 5)\}$ and $\mathcal{I}_3=\{(2, 1), (1, 4), (4, 2), (2, 3), (5, 2), (3, 1), (5, 1)\}$ are respectively a covering of Q.

Definition 3. Given a set of revealed weight ratios $\{a_{ij}\}$ over the pair index set $\mathcal{I}$, we define its *derived reciprocal matrix* $\hat{A}=[\hat{a}_{ij}]_{q \times q}$ by

$$\hat{a}_{ij} = \begin{cases} a_{ij} & \text{for each } (i, j) \in \mathcal{I}, i \neq j \\ 1/a_{ji} & \text{for each } (i, j) \notin \mathcal{I} \text{ but } (j, i) \in \mathcal{I} \\ 0 & \text{for each } (i, j) \text{ such that } (i, j) \notin \mathcal{I} \text{ and } (j, i) \notin \mathcal{I} \\ N_i & \text{for each } (i, i), \text{where } N_i \text{ is 1 plus the number of 0 in row i.} \end{cases}$$

Example 3. Suppose that a set of the revealed weight ratios are given as the following.

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & & 7 & & & 2 \\
2 & & & 3 & & \\
3 & & & & 4 & \\
4 & 8 & 2 & & & \\
5 & & 5 & & 3 &
\end{array}
$$

We have the corresponding derived reciprocal matrix.

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 2 & 7 & 0 & 1/8 & 2 \\
2 & 1/7 & 2 & 3 & 0 & 1/5 \\
3 & 0 & 1/3 & 3 & 4 & 0 \\
4 & 8 & 2 & 1/4 & 1 & 1/3 \\
5 & 1/2 & 5 & 0 & 3 & 2
\end{array}
$$

By inserting $\widetilde{w}_{ij}=\widetilde{w}_i/\widetilde{w}_j$ for the missing data in the revealed weight ratios, we obtain

<u>Definition 4</u>. Given a set of revealed weight ratios $\{a_{ij}\}$ over the pair index set $\mathscr{I}$, we define its *surrogated reciprocal matrix*, $B=[b_{ij}]_{q\times q}$ by

$$
b_{ij} = \begin{cases} a_{ij} & \text{for each } (i,j) \in \mathscr{I},\ i \neq j; \\ 1/a_{ji} & \text{for each } (i,j) \notin \mathscr{I}, \text{ but } (j,i) \in \mathscr{I}; \\ \widetilde{w}_{ij}=\widetilde{w}_i/\widetilde{w}_j & \text{for each } (i,j) \text{ such that } (i,j) \notin \mathscr{I} \\ & \quad \text{and } (j,i) \notin \mathscr{I}; \\ 1 & \text{for each } (i,i),\ i=1,2,\ldots,q \end{cases}
$$

The following theorem shows the intimate relationship between the derived and surrogated reciprocal matrices.

<u>Theorem 1</u>[4] Let $\hat{A}$ and B be respectively the <u>derived</u> and <u>surrogated</u> <u>reciprocal</u> <u>matrix</u> of a set of revealed weight ratios $\{a_{ij}\}$ over $\mathscr{I}$. Then $\hat{A}$ and B have identical eigenvalues and the corresponding eigenvectors.

The following theorem states, roughly, that if the revealed weight ratios are "consistent" over a covering $\mathscr{I}$ of Q, then the eigen weight vector obtained from the derived reciprocal matrix is the "true" weight vector

Theorem 2[4]. Given any consistent reciprocal matrix $W=(w_{ij})_{q\times q}$, let $\hat{w}$ be the eigenvector corresponding to the maximum eigenvalue $\lambda_{max}=q$ of W. For any covering $\mathcal{G}$ of Q, define a set of weight ratios as

$$a_{ij}=w_{ij} \text{ if } (i,j) \in \mathcal{G}.$$

Let $\hat{A}=(\hat{a}_{ij})_{q\times q}$ be the derived reciprocal matrix obtained from $\{a_{ij}\}$. Then

$$\tilde{\lambda}_{max}=q$$

and

$$\tilde{w} = \hat{w},$$

where $\tilde{\lambda}_{max}$ is the maximum eigenvalue of $\hat{A}$ and $\tilde{w}$ is the corresponding eigenvector.

Corollary[4] Given a set of the revealed weight ratios $\{a_{ij}\}$ over $\mathcal{G}$, assume that $\mathcal{G}$ is a covering of Q and the reciprocal matrix obtained by filling the missing data by transitivity and reciprocality is consistent. Then, the eigenvector $\tilde{w}$ associated with the maximum eigenvalue of the derived reciprocal matrix $\hat{A}$ coincides with that of the consistent reciprocal matrix.

Theorem 3[4]. For a given set of the revealed weight ratios $\{a_{ij}\}$ over $\mathcal{G}$, assume that $\mathcal{G}$ is a covering of Q. Then,

$$\lambda_{max} \geqq q, \text{ the equality holds iff } a_{ik}\times a_{kj} = a_{ij} \text{ for } (i,k), (k,j), \text{ and } (i,j) \in \mathcal{G}.$$

The following example illustrates that the procedure proposed here is useful for checking inconsistency in the course of pairwise comparisons interactively.

Example 4 Suppose that we have a set of the revealed weight ratios,

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & & 2 & 2 & & 9 \\ 2 & & & & 9 & \\ 3 & & & & & \\ 4 & & & & & 2 \\ 5 & & & & & \end{array}$$

Then we have

$\lambda_{max}=5.10$

$\hat{W}=(.420\ .308\ .191\ .048\ .034)$.

Suppose that we have an additional judgment, $a_{52}=5$. Then

$\lambda_{max}=7.483$

$\hat{W}=(.433\ .205\ .062\ .096\ .204)$.

Note that λ_{max} is too high. A careful study reveals that $\mathcal{F}$ contains a cycle, i.e., (2,4), (4,5), (5,2). This reveals that an adjustment is needed. Let us assume that we adjust it by setting $a_{25}=5$ instead of $a_{52}=5$. Then we obtain

$\lambda_{max}=5.14$

$\hat{W}=(.425\ .291\ .186\ .055\ .042)$

Finally suppose that we have all pairwise comparisons as follows.

	1	2	3	4	5
1		2	2	7	9
2			3	9	5
3				2	3
4					2
5					

The revealed reciprocal matrix A is a complete one.

	1	2	3	4	5
1	1	2	2	7	9
2	1/2	1	3	9	5
3	1/2	1/3	1	2	3
4	1/7	1/9	1/2	1	2
5	1/9	1/5	1/3	1/2	1

$\lambda_{max}=5.20$

$\hat{W}=(.422\ .333\ .138\ .061\ .045)$

4. Simulation Results

In this section, we present the simulation results on the comparisons between complete and incomplete reciprocal matrces in several cases. In each case, the number of replications is 500. First, the true weights w=(.3 .25 .2 .15 .1) are given. The statistical model is

$$a_{ij} = w_{ij} \times \varepsilon_{ij}$$

where $w_{ij}=w_i/w_j$ and the ε_{ij} are random variables with a mean of one. For our simulations we chose uniform distributions

Table 1. $w = [\,.3\ .25\ .2\ .15\ .1]$

$$D[i,j] = \begin{bmatrix} - & .5 & .5 & .5 & .5 \\ & - & .5 & .5 & .5 \\ & & - & .5 & .5 \\ & & & - & .5 \\ & & & & - \end{bmatrix}$$

$\mathscr{C}$ = {[i,j] | i < j }, all pairwise comparisons,

$\mathscr{S}_1$ = {[1,2], [1,3], [2,4], [3,5], [4,5] }

$\mathscr{S}_2$ = {[1,2], [1,3], [1,4], [1,5]}

true weights	[.300	.250	.200	.150	.100]	d_1
$\mathscr{C}$	[.289	.250	.201	.155	.105]	.512
$\mathscr{S}_1$	[.286	.250	.203	.154	.107]	.731
$\mathscr{S}_2$	[.282	.256	.203	.158	.102]	1.038

Table 2. $w = [\,.3\ .25\ .2\ .15\ .1]$

$$D[i,j] = \begin{bmatrix} - & .6 & .5 & .8 & .6 \\ & - & .8 & .6 & .4 \\ & & - & .3 & .4 \\ & & & - & .3 \\ & & & & - \end{bmatrix}$$

$\mathscr{S}$ = $\mathscr{C}$ − {[1,4], [2,3] }

true weights	[.300	.250	.200	.150	.100]	d_1
$\mathscr{C}$	[.285	.245	.209	.158	.102]	.643
$\mathscr{S}$	[.288	.252	.201	.155	.104]	.590

Table 3. $w = [\,.3\ .25\ .2\ .15\ .1]$

$$D[i,j] = \begin{bmatrix} - & .3 & .4 & .2 & .7 \\ & - & .3 & .4 & .6 \\ & & - & .3 & .8 \\ & & & - & .6 \\ & & & & - \end{bmatrix}$$

$\mathscr{S}$ = $\mathscr{C}$ − {[1,5], [3,5] }

true weights	[.300	.250	.200-	.150	.100]	d_1
$\mathscr{C}$	[.293	.246	.198	.150	.100]	.556
$\mathscr{S}$	[.292	.248	.200	.151	.110]	.528

Table 4. $w = [\,.3\ .25\ .2\ .15\ .1]$

$$D[i,j] = \begin{bmatrix} - & .3 & .2 & .6 & .3 \\ & - & .7 & .3 & .2 \\ & & - & .4 & .3 \\ & & & - & .4 \\ & & & & - \end{bmatrix}$$

$\mathscr{S}$ = $\mathscr{C}$ − {[1,4], [2,3] }

true weights	[.300	.250	.200	.150	.100]	d_1
$\mathscr{C}$	[.294	.246	.204	.154	.101]	.410
$\mathscr{S}$	[.297	.250	.199	.152	.102]	.351

for ε_{ij}. Second, the interval half-widths D(i,j) of ε_{ij}(Thus, ε_{ij} is bounded by $1 \pm D(i,j)$) are given in Tables 1-4.
In each case, summary averages are reported based on 500 replications. As a measure of 'goodness of fit' we adopted

$$d_1 = (1/N) \sum_{h=1}^{N} \sum_{i=1}^{5} | w_i - w_i^{(h)} | / w_i$$

where N=500 and $w_i^{(h)}$ is an estimate of w_i at h-th iteration.
Table 1 indicates that when the confidence of each pairwise judgment is almost the same, on average, completing all entries in the reciprocal matrix is most desirable. The more

the number of judgments answered, the more the estimated weights are close to the true weights.

Tables 2-4, however, suggest that if the low confident judgments are skipped, on average incomplete derived reciprocal matrices produce more favorable results.

5. Concluding Remarks

We have discussed a method to obtain an eigen weight vector based on incomplete pairwise weight ratios. The rationale and examples are also given. Simulation results suggest that (i) when the confidence of each pairwise comparison is nearly the same, completing all entries in the reciprocal matrix is best, (ii) when the confidence between pairwise comparisons are fairly different, skipping the low confident comparisons produces preferable results.

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References

[1] Cogger, K. O. and Yu, P. L. (1985), "Eigen weight vectors and least distance approxiation for revealed preference in pairwise weight ratios, Jour. Opt. Th. Appl., vol. 36.

[2] Saaty, T. L. (1980) The Analytic Hierarchy Process, McGraw-Hill, New York.

[3] Saaty, T. L., and Vargas, L. G. (1982) The Logic of Priorities, Applications in Business, Energy, Health, and Transportation, Kluwer-Nijhoff, Boston.

[4] Takeda, E., and Yu, P. L. (1988), "Eigen weight vectors from incomplete revealed weight ratios", presented at Ⅷ th conference on MCDM.

[5] Yu, P. L. (1985) Multiple-Criteria Decision Making, Concepts, Techniques, and Extensions, Prenum Press, New York.

[6] Zahedi, F. (1986), "The analytic hierarchy process-A survey of the method and its applications, Interfaces, vol. 16, pp 96-108.