SOME NOTES ON PAIRWISE COMPARISON MATRICES

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ABSTRACT

The existence of a best approximation to the pairwise comparison matrix from the set of consistent matrices is proved. At the same time, we can also prove that there are many best approximations. Then by diffeomorphism, we transform the primary nonlinear approximation into a linear one. Hence we get an approximate method to abtain the best approximation. Finally, we give a simple example.

1. Introduction

The most important and exciting theory in AHP is that when the matrices obtained by the expert's are consistent, the eigenvector corresponding to the eigenvalue with the largest modulus is exactly the one which represents the priority of factors compared; and it can be obtained by normalizing the column of the matrix. Therefore, it is possible to make the calculation simple and reliable. However, in practical application, when the compared factors in the same hierarchy are more than three, it is dfficult for the experts to measure their priority with the same scale, consequently, the obtained matrices are often nonconsistent. It is no doubt it sets obstacles in our practical application. How can we do if we want to make full use of good properties of the consistent matrix as well as to give a thoughtful consideration of the experts'opinion ? we have to adapt a balanced way, that is, to correct the pairwise comparison matrices obtained by the experts by an amount as minimum as possible, thus making them become consistent. In other words, we'll find the best approximation to the pairwise comparison matrix from the set of consistent matrices. To begin with, this paper theoretically gives a proof of the existence of this best approximation, at the same time, we can also prove that there are many best approximations for our problem. Then by diffeomorphism, we transform the primary nonlinear approximation into a linear one. Hence we get an approximate method to abtain the best approximation. Finally, we give some simple examples and analysis of the error and consequence obtained.

2. Statement of the Basic Problem

Let $\mathbf{R}^{n\times n}$ be a linear space consisted of all the real nxn matrices. The set of positive reciprocal matrices and the set of the positive consistent matrices are denoted by $\mathbf{P}$ and $\mathbf{Q}$, that is,

$$\mathbf{P}=\{p=(p_{ij})\in\mathbf{R}^{n\times n}\mid p_{ij}>0,\ p_{ik}=p_{ki}^{-1}\},$$

$$\mathcal{Q}=\left\{q=(q_{ij})\in \mathcal{R}^{n\times n} \mid q_{ij}>0,\quad q_{ik}q_{ki}=q_{ij}\right\}.$$

Obviously, both $\mathcal{P}$ and $\mathcal{Q}$ are not the convex subset of $\mathcal{R}^{n\times n}$.
If we definite the norm for $a=(a_{ij})\in \mathcal{R}^{n\times n}$ as

$$\|a\|_p=\Big(\sum_{ij}|a_{ij}|^p\Big)^{\frac{1}{p}} \qquad 1\leq p\leq +\infty$$

then we obtained a normed linear space $(\mathcal{R}^{n\times n},\|\cdot\|_p)$. Various norms on finite dimensional space are equivalent, in other words, leading to the same topology. Nevertheless, we have different Geometries for $p=1$, $+\infty$ or $p\in(1,+\infty)$. Specially, for the simpleness and feasibility in mathematics, we would rather let $p=2$, then, the normed space $(\mathcal{R}^{n\times n},\|\cdot\|_2)$ is a n^2-dimension Euclidean space, inner product and norm in which are

$$(a,b)=t_r(b^Ta),\quad a,b\in \mathcal{R}^{n\times n}$$

$$\|a\|_2=t_r(a^Ta)^{\frac{1}{2}}=\Big(\sum_{ij}|a_{ij}|^2\Big)^{\frac{1}{2}}.$$

For convenience, we, represent $\|\cdot\|$ for $\|\cdot\|_2$.
Let M be a non-empty subset of $\mathcal{R}^{n\times n}$. Given $a\in \mathcal{R}^{n\times n}$, we definite the distance a to M as

$$d(a,M)=\inf_{m\ M}\|a-m\|.$$

If $m_0\in M$ satisfies $\|a-m_0\|=d(a,M)$, then m_0 is said to be a best approximation or a nearest point to a from M.
Our problem is : qiven $p_0\in\mathcal{P}$, we require the best approximation to p_0 from $\mathcal{Q}$, that is ,find $q_0\in\mathcal{Q}$ such that

$$\|p_0-q_0\|=d(p_0,\mathcal{Q})=\inf_{q\in\mathcal{Q}}\|p_0-q\|.$$

3. The Main results

PROPOSITION 1. Given $p_0\in\mathcal{P}$, there exists at least a $q_0\in\mathcal{Q}$, such that

$$\|p_0-q_0\|=d(p_0,\mathcal{Q})=\inf_{q\in\mathcal{Q}}\|p_0-q\|.$$

Proof. Let $\delta=d(p_0,\mathcal{Q})$. In normed space $\mathcal{R}^{n\times n}$, the convergence of the matrix-sequence is equivalent to convergence of component-wise of matrices. Obviously, $\mathcal{Q}$ is a closed set in $\mathcal{R}^{n\times n}$. By definition of infimum, there exists sequence $q_n\in\mathcal{Q}$ such that

$$\delta_n=\|p_0-q_n\|\longrightarrow\delta\ (n\rightarrow\infty).$$

It implies sequence $\{q_n\}$ is bounded. By Bolzano-Weierstrass theorem, at least there is a subsequence of q_{m_k}, such that

$$\lim_{n_k\to\infty} q_{n_k}=q_0.$$

Then by the closeness of $\mathcal{Q}$, we have $q_0\in\mathcal{Q}$. By the continuity of norm, we have

$$\delta=\lim_{n_k\to\infty}\delta_{n_k}=\lim_{n_k\to\infty}\|p_0-q_{n_k}\|=\|p_0-\lim_{n_k\to\infty}q_{n_k}\|=\|p_0-q_0\|.$$

Thus we finish the proof of the proposition 1.
About the uniqueness of the best approximation, we quote the following result directly [1].

LEMMMA 2. In a smooth and strict convex finite dimensional space, the following conditions are equivalent:

(i) M is closed and convex;
(ii) M is a chebyshev set ;
(iii) M is a sun .

PROPOSITION 3. Given $p_0\in\mathcal{P}$, $q_0\in\mathcal{Q}$ which satisfies $\|p_0-q_0\|=d(p_0,\mathcal{Q})$ is not unique.

Proof. A normed space X is called smooth, if at each point $x\in S$ S (S is the unit sphere in X) there exists a unique supporting hyperplane, or equivalently, there exists a unique peak functional to $x\neq 0$. If X is reflexive, then the smoothness of X is equivalent to the strict convexity of X′ (X′ is the dual space of X). we know all finite dimensional spaces are reflexive, so we only need prove $\mathcal{R}^{n\times n}$ is strict convex. By the clarkson's inequality

$$\|a+b\|_p^{\frac{p}{p-1}}+\|a-b\|_p^{\frac{p}{p-1}}\leqslant 2\left(\|a\|_p^p+\|b\|_p^p\right)^{\frac{p}{p-1}},\qquad 1<p\leqslant 2$$

obviously, we have for p=2, $\|a\|_2=\|b\|_2=1$, and $a\neq b$:

$$\|a+b\|_2^2\leqslant 2^2-\|a-b\|_2^2<4$$

i.e.

$$\|a+b\|_2<2\quad .$$

By definition, we know $(\mathcal{R}^{n\times n},\|\cdot\|_2)$ is strict convex. Therefore, it is smooth. Since $\mathcal{Q}$ is not convex, but it is closed, by lemma 2, $\mathcal{Q}$ is not chebyshev set. Then by proposition 1, we Know there exist more than one best approximations to p_0 from $\mathcal{Q}$.

As our problem is a nonlinear approximation, it is difficult for to describe the common characteristic of the best approximations. Further, since there exist many best approximations, we have hard time to construct a common calculated method. However, for some specific nonlinear approximations, we can construct a best approximation. For example, by the singular-value decomposition of matrix, we can construct a r-rank matrix and orthogonal matrix to approximate it. Here by diffeomorphism we transform a nonlinear approximation into a linear approximation. But, as the diffeomorphism is not isometric, the former and the latter approximations are not equivalent. After all, there is a method to solve it simply

and feasibly.
The matrices of $\mathcal{P}$, $\mathcal{Q}$ are all positive, so we can definite map

$$f: \mathcal{P} \longrightarrow \mathcal{R}^{n\times n}$$
$$p=(p_{ij}) \longmapsto f(p)=(\ln p_{ij}).$$

PROPOSITION 4. For sets $\mathcal{P}$ and $\mathcal{Q}$, obviously, we have

$$f(\mathcal{P})=\left\{a=(a_{ij})\in \mathcal{R}^{n\times n} \,\middle|\, a^T=-a\right\}, \qquad (1)$$
$$f(\mathcal{Q})=\left\{b=(b_{ij})\in \mathcal{R}^{n\times n} \,\middle|\, b_{kj}=b_{1j}-b_{1k}, \quad 1\leq k,\ j\leq n\right\} \qquad (2)$$

this implies $f(\mathcal{P})$ and $f(\mathcal{Q})$ are the subspaces of $\mathcal{R}^{n\times n}$. For $f(\mathcal{Q})$, we can choose a basis

$$\left\{ E_{1i}=\sum_{\substack{k=1\\k=j}}^{n}(e_k e_i^T-e_i e_k^T), \qquad 2\leq i\leq n \right\}$$

where $e_i \in \mathcal{R}^n$, its i-th component is one, the other is vanished.
Proof. (1) is obvious. For any $q=(q_{ij})\in \mathcal{Q}$ we have $q_{ij}=q_{ik}q_{kj}$. Particularly, let i=1, then

$$q_{kj}=q_{1j}/q_{1k}, \qquad 1\leq k,\quad j\leq n.$$

Thus $\ln q_{kj}=\ln q_{1j}-\ln q_{1k}$. Let $f(q)=(\ln q_{ij})=b=(b_{ij})$, so $b_{kj}=b_{1j}-b_{1k}$. this indicates

$$b=(b_{ij})=\begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} & \cdots\cdots & b_{1n} \\ -b_{12} & 0 & b_{13}-b_{12} & b_{14}-b_{12} & \cdots\cdots & b_{1n}-b_{12} \\ -b_{13} & b_{12}-b_{13} & 0 & b_{14}-b_{13} & \cdots\cdots & b_{1n}-b_{13} \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ -b_{1n} & b_{12}-b_{1n} & b_{13}-b_{1n} & b_{14}-b_{1n} & \cdots\cdots & 0 \end{pmatrix}$$
$$=b_{12}E_{12}+b_{13}E_{13}+\cdots\cdots+b_{1n}E_{1n}.$$

This implies every matrix in $f(\mathcal{Q})$ can linearly be represented by $\{E_{1i}\}_{2\leq i\leq n}$. Obviously $\{E_{1i}\}_{2\leq i\leq n}$ is linear independent, thus $\{E_{1i}\}_{2\leq i\leq n}$ is a basis of $f(\mathcal{Q})$, and $\dim f(\mathcal{Q})=n-1$.
PROPOSITION 5. Given $p_0=(p_{ij})\in\mathcal{P}$, then for $\tilde{p}_0=(\ln p_{ij})\in f(\mathcal{P})$, there exists unique best approximation b_0 to $\tilde{p}_0$ from $f(\mathcal{Q})$. If we let $b_0=\sum_{i=1}^{n}\lambda_i E_{1i}$, then $\lambda=(\lambda_2,\lambda_3,\cdots,\lambda_n)^T$ satisfies linear matrix equation

$$G\lambda = \alpha$$

where

$$G=\begin{pmatrix} (E_{12},E_{12}) & (E_{13},E_{12}) & \cdots\cdots & (E_{1n},E_{12}) \\ (E_{12},E_{13}) & (E_{13},E_{13}) & \cdots\cdots & (E_{1n},E_{13}) \\ & \vdots & & \\ (E_{12},E_{1n}) & (E_{13},E_{1n}) & \cdots\cdots & (E_{1n},E_{1n}) \end{pmatrix} \quad (n-1)\times(n-1)$$

$$\alpha = ((\tilde{p}_0,E_{12}),\ (\tilde{p}_0,E_{13}),\ \ldots\ldots,\ (\tilde{p}_0,E_{1n}))^T$$

so $\lambda = G^{-1}\alpha$ and $b_0=(E_{12},E_{13},\ \ldots\ldots,\ E_{1n})G^{-1}\alpha$.

Proof. by the projection theorem in inner product space, we can easily draw this conclusion (with reference to [2]).
By the proposition 5, we can obtain the best approximation b_0 to $\tilde{p}_0$ from $f(Q)$. Then let $q_0=f^{-1}(b_0)$, thus we attain a consistent matrix q_0, regard q_0 as a best approximation to p_0.

4. An example and analysis

Here we give an example for $p_0\in R^{4\times4}$, also we compare our results with the consequences obtained by other methods.

$$p_0=\begin{pmatrix} 1 & 2.2 & 3.2 & 3.9 \\ 0.4545 & 1 & 1.8 & 2.4 \\ 0.3125 & 0.5556 & 1 & 1.2 \\ 0.2564 & 0.4167 & 0.8333 & 1 \end{pmatrix}$$

$$\ln p_0=\tilde{p}_0=\begin{pmatrix} 0 & 0.7885 & 1.1632 & 1.3610 \\ -0.7885 & 0 & 0.5878 & 0.8755 \\ -1.1632 & -0.5878 & 0 & 0.1823 \\ -1.3610 & -0.8755 & -0.1823 & 0 \end{pmatrix}$$

$$E_{12}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\quad E_{13}=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$E_{14}=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

$(\tilde{p}_0,E_{12})=t_r(E_{12}^T\tilde{p}_0)=-1.3556$, $(\tilde{p}_0,E_{13})=3.1374$, $(\tilde{p}_0,E_{14})=4.8376$,

$\alpha=(-1.3556,\ 3.1374,\ 4.8376)^T$. $(E_{12},E_{12})=t_r(E_{12}^TE_{12})=6$
$=(E_{13},E_{13})=(E_{14},E_{14})$, $(E_{12},E_{13})=(E_{12},E_{14})=-2$.

$$G=\begin{bmatrix} 6 & -2 & -2 \\ -2 & 6 & -2 \\ -2 & -2 & 6 \end{bmatrix}, \quad G^{-1}=\frac{1}{8}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\lambda=G^{-1}\alpha=(0.658,\ 1.2196,\ 1.4321)^T$$

$$b_0=0.658E_{12}+1.2196E_{13}+1.4321E_{14}$$

$$=\begin{bmatrix} 0 & 0.6580 & 1.2196 & 1.4321 \\ -0.6580 & 0 & 0.5616 & 0.7741 \\ -1.2196 & -0.5616 & 0 & 0.2125 \\ -1.4321 & -0.7741 & -0.2125 & 0 \end{bmatrix}$$

$$q_0=f^{-1}(b_0)=\begin{bmatrix} 1 & 1.9309 & 3.3858 & 4.1875 \\ 0.5179 & 1 & 1.7535 & 2.1686 \\ 0.2954 & 0.5703 & 1 & 1.2368 \\ 0.2388 & 0.4611 & 0.8085 & 1 \end{bmatrix}$$

Acknowledgment

The authors take this opportunity to express their gratitude to Dr Ma Zhi-Wei for a number of valuable suggestions and useful conversation.

Reference

[1] braess, D. (1986) 《Nonlinear Approximation Theory》, p41, Springer-Verlag Berlin Heidelberg.
[2] Luenberger, D.G. (1969)《Optimization by Vector Space Methods》, John Wiley & Sons, Inc.