ORDINAL AND CARDINAL PAIRWISE COMPARISONS IN AHP

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ABSTRACT

In the AHP, the decisionmakers must construct a set of pairwise comparison matrices. In this procedure, the judgments of pairwise comparisons are, generally, expressed as the cardinal numbers, such as the one-to-nine scale of relative importance. This is so-called cardinal pairwise comparisons. But there often exist some difficulties in deriving these judgments of the cardinal pairwise comparisons. 1). The understand and master of the cardinal scale are difficult for the decisionmakers to use, especially in China; 2). The judgments could often be conjectural, casual, and fuzzy; 3). The cardinal scale would make the questionaire in AHP difficult; 4). the complex problems could need much time in use of the cardinal scale. In order to overcome these difficulties, the concepts and approaches of the ordinal pairwise comparison matrices are presented in this paper. With the help of the judgments of the ordinal pairwise comparisons and one judgment of the cardinal pairwise comparison, the decisionmakers can easily apply the AHP. The applications and comparative analysis are given in the paper, which show that the improvement of the AHP is simple and easy in use, reliable and available in results, and good in consistency.

I. Introduction

The Analytic Hierarchy Process (AHP) provides the decisionmakers with a comprehensive framework and effective tool to solve the complex problems, which are to choose a best one or to rank everyone in a set of competing alternatives that are evaluated under multiple criteria, usually conflicting, criteria. And the problems for multiple criteria decision making (MCDM) are common occurrences in our real world, so the use of the AHP is widespread to solve these problems in many areas such as in political, economic, social, military, scientific and technological field and so on.

There are three procedures in AHP, which are the decomposition, comparative judgments, and synthesis of priorities. The comparative judgments are the critical procedure, in which the decisionmakers must carry out a series of pairwise comparisons of relative importances of the elements in lower level with respect to overall objective (or focus) of the above level in the hierarchy, constructed in the procedure of the decomposition, to set up the judgment matrices. These judgments of the pairwise comparisons are generally given with the cardinal scales of measurement. A cardinal scale, the one-to-nine scale, is presented (Saaty, 1980, 1983, 1986). This cardinal scale is compared by Saaty with twenty-eight other different scale through gathering and calculating considerable experimental data, and the evidence strongly favors the use of the one-to-nine scale as a reflection of our mental ability to discriminate different degrees of strengths of dominance among the objects (Saaty, 1980).

But there often exist some difficulties in using the cardinal scale to derive judgment information, as Saaty said: Most of the difficulties encountered in the use of AHP related the need for judgments (Saaty, 1986). These difficulties include:

1. The decisionmakers must understand and master the cardinal scale of relative importance, otherwise they can not easily, conveniently, and correctly present their judgments with the cardinal scale. And the understanding and mastering are difficult to them, especially in China;

2. The judgments of the cardinal pairwise comparisons given by the decisionmakers could often be conjectural, casual and fuzzy, so the reliability of the judgments could be suspected, and the judgments may also result in lack of consistency of the cardinal pairwise comparisons matrices, especially of the higher order matrices;

3. The use of the cardinal scale would also result in problem of questionaire in AHP (the difficulties of design and fill of the questionaire). So it is difficult to collect the judgment information of multiple decisionmakers (group decision making) or of Delphi;

4. If a problem is complex and requires careful analysis, then much time could be needed to elicit the judgments (Saaty,1986).

In order to overcome these difficulties, in this paper, the concepts of the ordinal pairwise comparisons and indirect cardinal judgments are presented. And the approaches of deriving indirect cardinal judgment matrices are given. The use and the compare of the concepts and approaches, which are introduced in AHP, demonstrate that the above difficulties canbe wellovercome.

II. Concepts and Approaches

When the decisionmakers compare two elements, A_i and A_j, with respect to criterion above, they can easily, conveniently and correctly present the following judgment: which is more important, or less important, or equal important. If the ordinal scale is defined as follows:

$$c_{ij} = \begin{cases} 0, & \text{if } A_i \text{ is less important than } A_j \\ 1, & \text{if } A_i \text{ is equal important as } A_j \\ 2, & \text{if } A_i \text{ is more important than } A_j \end{cases}$$

the comparative judgments with this scale are called as the ordinal pairwise comparisons. The judgements of the ordinal pairwise comparsons of n elements can be expressed as the following ordinal judgment matrix:

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

The c_{ij} can be derived with the help of the following tableau format (Fig. 1)

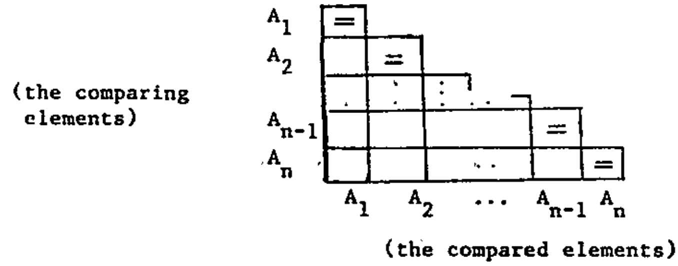

Fig. 1. Tableau format of deriving c_{ij}

The lattices in tableau are filled with the notation of "<", "="or ">", by decisionmakers according to the ordinal pairwise comparisons. The "<", "=" and ">" express, respectively, less important, equal important, and more important between A_i and A_j.

Although the judgments of the ordinal pairwise comparisons are simply and easily derived, there is no information of the intensity (strength) of relative importance between two elements, and the ratio relation among the elements can not be calculated by means of the ordinal information.

But if a so-called index of rank dominance is defined as follows:

$$r_i = \sum_{j=1}^{n} c_{ij} \qquad (1)$$

where r_i: the index of rank dominance of A_i

c_{ij}: the ordinal judgment information described above.

the relative relation between two elememts can be determined according to r_i and r_j, which is the distance of the rank dominance between A_i and A_j. In other hand, if a bit of the cardinal information are introduced in the relative relations among elements, the ratio relations of the elements can be elicited.

Let $r^* = \max(r_i)$, $r^\circ = \min(r_i)$, $A^* = (A_i \mid \max(r_i))$, $A^\circ = (A_i \mid \min(r_i))$, and compare the reletive importance of element A^* and element A° with the cardinal scales, e.g. the one-to-nine, the information of intensity of relative importance, by the name of b_m, is given. So we can use the following formulae to obtain the cardinal information of the pairwise comparisons among elements:

$$b_{ij} = \begin{cases} (b_m-1)[(r_i-r_j)/(r^*-r^\circ)] + 1, & \text{if } r_i-r_j \geq 0 \quad (2) \\ [(b_m-1)[(r_j-r_i)/(r^*-r^\circ)] + 1]^{-1} & \text{if } r_i-r_j < 0 \quad (3) \end{cases}$$

where b_{ij} is the indirect cardinal judgment of pairwise comparison between A_i and A_j; r_i-r_j is the distance of rank dominance between A_i and A_j; b_m is the cardinal judgment of pairwise comparison between A^* and A°.

Here the b_m is called as "base point" of the indirect cardinal judgments, and the matrix, $B = [b_{ij}]_{nxn}$, is called the indirect cardinal judgment matrix. The b_{ij} satisfies:

1) $1/b_m \leqslant b_{ij} < 1,$ if $b_{ij} < 1$ (4)

$1 \leqslant b_{ij} \leqslant b_m,$ if $b_{ij} \geqslant 1$ (5)

2) $b_{ij} = 1/b_{ji}$ (6)

that is, the b_{ij} satisfies the homogeneous axiom and reciprocal axiom in AHP (Saaty, 1986).

Based upon the indirect matrix, the weights of the elements, and the ratio relations among the elements, can be calculated by means of the Eigenvecter Method(EM), or Least Square Method(LSM), or Logarithm Least Square Method(LLSM), etc. (Fichtner, 1986; Zahedi, 1986). By the way, if the bm is determined in the one-to-nine scale , the entries of the indirect judgment matrix will be nearly the same as the entries in the matrix constructed according to Saaty's rules.

The following demonstrations include the applications of the concepts and approaches of the ordinal pairwise comparisons and the indirect cardinal judgments, and the comparative analysis of the ordinal and cardinal pairwise comparisons.

III. Application and Comparative Analysis

1. Application of evaluating relative level of teaching

In order to determine the relative teaching levels of the teachers in author's department, the AHP with the indirect cardinal pairwise comparisons is applied. The hierarchy of the evaluation of the relative teaching level is given in Fig. 2. Here only four teachers are considered to be evaluated to spare space.

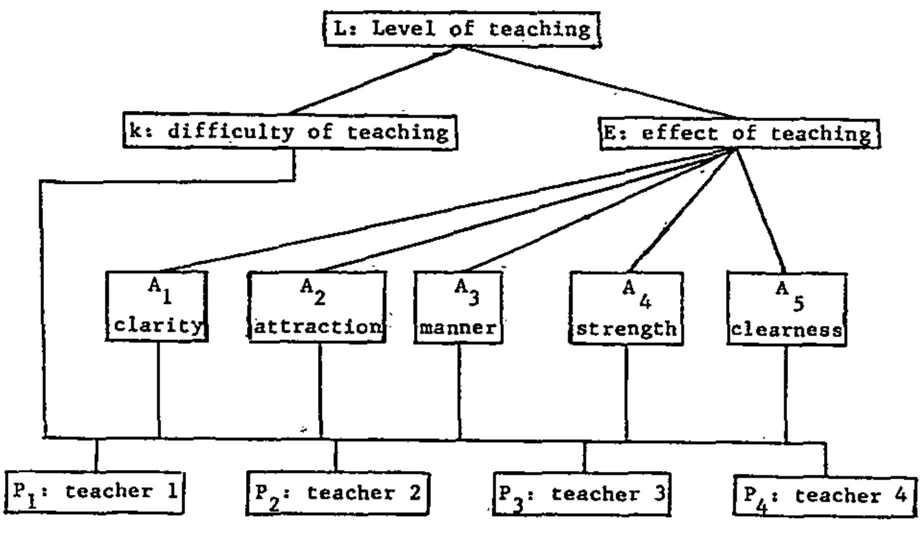

Fig. 2. A hierarchy of evaluating the teaching level

The relative teaching level of a teacher can be measured by

$$L = kE \qquad (7)$$

where L: is the relative teaching level of the teacher; E is the relative effect of the teaching; k is the relative degree of the difficulty of the teaching content.

The effect of the teaching consists of five factors (criteria): A_1 is the clarity of the teaching; A_2 is the attraction of the teaching language; A_3 is the manner of the teacher in teaching; A_4 is the strength of the speak in teaching; A_5 is the clearness of the writing on the blackboard.

The ordinal parwise comparison matrix among the above factors and the "base point" judgment between A^* and A° are given in Table 1, and the indirect cardinal judgment matrix is presented in Table 2.

Table 1 The matrix of ordinal judgments

E	A_1	A_2	A_3	A_4	A_5	w_i	
A_1	1	2	2	2	2	9	$r^*=9$
A_2	0	1	2	2	2	7	$r^\circ=1$
A_3	0	0	1	0	0	1	$A^*=A_1$
A_4	0	0	2	1	2	5	$A^\circ=A_3$
A_5	0	0	2	0	1	3	$A_1:A_3 \Rightarrow b_m=7$

Table 2. The matrix of the indirect cardinal judgments

E	A_1	A_2	A_3	A_4	A_5	w_1	
A_1	1	2.5	7	4	5.5	.471	
A_2	1/2.5	1	5.5	2.5	4	.265	$\lambda_{max}=5.146$
A_3	1/7	1/5.5	1	1/4	1/2.5	.044	C.I.=0.037
A_4	1/4	1/2.5	4	1	2.5	.143	C.R.=0.033
A_5	1/5.5	1/4	2.5	1/2.5	1	.077	

With respect to A_i (i=1,2,...,5), four teachers are compared pairwise to obtain five ordinal judgement matrices and indirect cardinal judgement matrices (They are omitted in the paper to spare space). The results of calculating the indirect cardinal judgement matrices are shown as follows (E_j is the relative teaching effect of each teacher, j=1,2,3,4):

	A_1	A_2	A_3	A_4	A_5	E_j
w (	.471	.265	.044	.143	.077)	
P_1	.143	.190	.274	.214	.497	.199
P_2	.506	.555	.506	.214	.111	.447
P_3	.274	.190	.143	.477	.280	275

P_4	.007	.065	.077	.095	.111	.079
λ_{max}	4.056	4.064	4.056	4.025	4.023	
C. I.	.019	.021	.019	.008	.008	
C. R.	.021	.024	.021	.009	.009	

The reletive difficulty of the teaching content of each teacher , k_j, can be determined with the help of Table 3 and Table 4.

Table 3 Matrix of ordinal

k	P_1	P_2	P_3	P_4	r_j
P_1	1	0	0	0	1
P_2	2	1	0	0	3
P_3	2	2	1	2	7
P_4	2	2	0	1	5

$r^*=7$, $r^o=1$

A^*: $A^o=P_3$: $P_1 \Rightarrow b_m=2$

Table 4 Matrix of indirect cardinal

k	P_1	P_2	P_3	P_4	k_j
P_1	1	3/4	1/2	3/5	.1657
P_2	4/3	1	3/5	3/4	.2117
P_3	2	5/3	1	4/3	.3493
P_4	5/3	4/3	3/4	1	.2733

$\lambda_{max}=4.002$

C. I. =0.001, C. R. =0.001

So the reletive teaching level of each teacher, L_j, can be measured:

$$L_1=k_1E_1=0.1657\times0.199=0.0330$$
$$L_2=k_2E_2=0.2117\times0.447=0.0946$$
$$L_3=k_3E_3=0.3493\times0.275=0.0961$$
$$L_4=k_4E_4=0.2733\times0.079=0.0216$$

They can be also normalized to L'_j:

$$L'_1=0.134,\quad L'_2=0.386,\quad L'_3=0.392,\quad L_4=0.088.$$

According to L'_j (or L_j), the teaching level of the teachers can be evaluated and classified.

2. Comparative analysis of examples of applying both ordinal and cardinal.

The author presented a series of comparative analysis of the examples of applying the ordinal and cardinal pairwise comparison (Zuo, 1985). To spare space, one example of choosing the best house to buy (Saaty, 1983), is given in this paper. The decomposition of the problem includes:In the first level is the overall goal of "satisfaction with house"; In the second level are the eight factors or criteria which contribute to the goal, and in the third level are the three candidate houses which are to be evaluated in terms of the criteria in second level (Saaty, 1983).

The weight vector of the eight criteria is calculated with the cardinal pairwise judgment matrix (Tab. 5)

Table 5 Pairwise comparison matrix for level 2

	1	2	3	4	5	6	7	8	weight vector	
1	1	5	3	7	6	6	1/3	1/4	.173	
2	1/5	1	1/3	5	3	3	1/5	1/7	.054	
3	1/3	3	1	6	3	4	6	1/5	.188	
4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8	.018	
5	1/6	1/3	1/3	3	1	1/2	1/5	1/6	.031	λ_{max}=9.669
6	1/6	1/3	1/4	4	2	1	1/5	1/6	.036	C.I.=0.238
7	3	5	1/6	7	5	5	1	1/2	.167	C.R.=0.169
8	4	7	5	8	6	6	2	1	.333	

Table 6 Matrix of ordinal judgments

	1	2	3	4	5	6	7	8	r	
1	1	2	2	2	2	2	0	0	11	
2	0	1	0	2	2	2	0	0	7	
3	0	2	1	2	2	2	2	0	11	
4	0	0	0	1	0	0	0	0	1	
5	0	0	0	2	1	0	0	0	3	
6	0	0	0	2	1	1	0	0	5	$r^*=15$
7	2	2	0	2	2	2	1	0	13	$r^\circ=1$
8	2	2	2	2	2	2	2	2	15	$A^*:A^\circ \Rightarrow b_m=8$

Table 7 Matrix of indirect cardinal judgments

	1	2	3	4	5	6	7	8	W	
1	1	3	1	6	5	4	1	1/3	.161	
2	1/3	1	1/3	4	3	2	1/3	1/5	.072	
3	1	3	1	6	5	4	1	1/3	.161	
4	1/6	1/4	1/6	1	1/2	1/3	1/6	1/8	.024	
5	1/5	1/3	1/5	2	1	1/2	1/5	1/7	.033	λ_{max}=8.246
6	1/4	1/2	1/4	3	2	1	1/4	1/6	.048	C.I.=0.035
7	1	3	1	6	5	4	1	1/3	.161	R.I.=0.025
8	3	5	3	8	7	6	3	1	.339	

Based on the above cardinal judgment matrix, the ordinal judgment matrix is given in Table 6, and indirect cardinal judgment matrix is presented in Table 7.

From this example we can see that the consistency of the direct cardinal judgment matrix is worse than the indirect cardinal judgment matrix, and the weight vectors are almost consistent.

IV Conclusion

The Analytic Hierarchy Process is an effective tool to solve complex MCDM problems in the real world. But there are some difficulties in applying the AHP for decisionmakers. In this paper the improvement of AHP is discussed on the pairwise comparison judgments. By means of the improvement, the ordinal pairwise comparisons, the decisionmakers can easier apply to the AHP. The applications and comparative analysis demonstrate that the improvement of the AHP is simple and easy in use, reliable and available in results, and good in consistency.

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