EXAMPLES OF DIFFICULTIES WITH ORDINAL PREFERENCE THAT DISAPPEAR WITH CARDINAL PREFERENCE

Thomas L. Saaty and Luis G. Vargas
University of Pittsburgh, Pittsburgh, PA 15260, USA.
Saaty@vms.cis.pitt.edu, Vargas@vms.cis.pitt.edu

Abstract: We examine some examples where the use of cardinal preferences solves problems that occur when ordinal preferences are used exclusively. Ordinal preferences are the first step toward ranking alternatives but they are not the last. In general we are interested in deriving a consistent ranking of alternatives which may then be used to construct a scale after a unit of measurement has been developed.

Introduction

Until recently, the theories used to make decisions have been based on ordinal preferences. Ordinal ranking of n alternatives is essentially derived by making comparisons of the alternatives in pairs. Only when transitivity holds can the elements be totally ordered. Otherwise, inconsistencies arise from using comparisons and one cannot have a total order. In general, we are interested in deriving a consistent ranking of the alternatives which may be used to create a scale once the unit has been agreed upon. Because numbers are totally ordered, one often associates numbers with ranks. Thus, when A is preferred to B, one may assign to A a number larger than that assigned to B. There is no loss of generality in assuming that these numbers are 1 and 0.

Ordinal preferences can be represented as a limiting case of cardinal pairwise comparisons. Consider two alternatives A and B. When we choose alternative A over alternative B, we are implicitly assigning more weight to A than to B. The question is how much more weight. Because ordinal preferences can be interpreted as (0,1) decisions, the most preferred alternative is assigned the value 1 and the least preferred one the value 0. This assignment can be easily represented by a numerical pairwise comparison a. It represents the intensity of preference of one alternative over the other. When a tends to infinity, the most preferred alternative is assumed to be infinitely more preferred than the least preferred one. Comparisons of two alternatives can be arranged in a 2¥2 matrix as follows:

$$\Omega = \begin{matrix} & A & B \\ A & 1 & a \\ B & 1/a & 1 \end{matrix} \qquad (1)$$

This matrix has the principal right eigenvector $\begin{pmatrix} \frac{a}{1+a} \\ \frac{1}{1+a} \end{pmatrix}$ representing the weights of A and B. As $a \rightarrow \infty$ this vector converges to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with weights 1 and 0 for A and B, respectively. In this manner one can show that with transitivity any ordinal ranking of n alternatives can be obtained as a limit of a cardinal ranking of these alternatives obtained as an ordered sequence of (n-1) paired comparisons such that two contiguous pairs have an element in common. The 1 and 0 obtained do not belong to a scale to rank all the alternatives but are only a representation of preference.

We show through examples that problems that have been presented in the literature as impossibilities or paradoxes because of the use of ordinal preferences can be explained by using cardinal pairwise comparisons.

Group Choice

Arrow's Impossibility theorem deals exclusively with ordinal comparisons. However, once the ordinal comparisons are changed to cardinal comparisons, the theorem is no longer an impossibility. This work is part of the doctoral dissertation of Kirti Peniwati (1996). Here we summarize some of her research.

Arrow's theorem requires that individual decision makers' preferences be a weak order, i.e., transitive and irreflexive. However, because pairwise reciprocal comparisons may satisfy consistency, a more general property than transitivity, it is possible to construct social functions in the same manner as Arrow does. It is noteworthy that when the paired comparisons tend to infinity, i.e., cardinal preferences become ordinal preferences, Arrow's Theorem obtains.

Consider a social function, $F: X \times D \rightarrow X$ where **X** is the set of alternatives, and **D** is set of decision makers **D** with the following conditions:

(i) the preferences of each decision maker are a weak order;
(ii) the set of alternatives **X** has cardinality ≥ 3;
(iii) every triplet of alternatives is free in the set of decision makers **D**, i.e., every possible m-tuple of individual's preferences on every triplet of alternatives, appears in some set of preferences corresponding to a set $D \subset \mathbf{D}$ of decision makers;

If F_D is an asymmetric binary relation on **X** associated with **F** such that when $A_i F_D A_j$ the alternative A_i is selected, then one of the following conditions must be false:

(iv) F_D is a weak order on every $D \subset \mathbf{D}$.
(v) if all the decision makers prefer A_i over A_j then $A_i F_D A_j$;
(vi) if two different groups of decision makers D and D' have the same preferences over $\{A_i, A_j\}$, then $F_D = F_{D'}$ on $\{A_i, A_j\}$;
(vii) there is no decision maker whose preferences coincide with the group preferences over the entire set of alternatives.

Let us now represent the decision makers' preferences by means of pairwise reciprocal comparisons.

Definition 1: A consistent order $\succ_k^c$ is a binary relation on the set of alternatives such that $A_i \succ_k^c A_j$ if and only if $a_k(i,j) > 1$, where $a_k(i,j)$ is the (i,j) entry of the consistent matrix formed with the pairwise comparison preferences of the kth decision maker.

Definition 2: A p-dominant order $\succ_k^p$ is a binary relation on the set of alternatives such that $A_i \succ_k^p A_j$ if and only if $a_k^p(i,h)/a_k^p(j,h) > 1$, where $a_k^p(i,h)$ is the (i,h) entry of the pth power of the reciprocal matrix formed with the pairwise comparison preferences of the kth decision maker.

Conditions (iv)-(vii) are consistent among themselves if weak order is substituted for consistent order and the social function is defined such that the asymmetric binary relation associated with it is given by:

$$A_i F_D A_j \Leftrightarrow \left[\prod_{k=1}^{m} a_k(i,j)\right]^{1/n} > 1.$$

Similarly, a consistent order can be substituted for a p-dominant order and the social function defined in terms of the binary relation given by:

$$A_i F_D A_j \Leftrightarrow \frac{\left[\prod_{k=1}^{m} a_k^p(i,h)\right]^{1/m}}{\left[\prod_{k=1}^{m} a_k^p(j,h)\right]^{1/m}} > 1$$

Utility Functions without the Use of Lotteries

In utility theory one must begin by ordering alternatives without measurement. This is then followed by the elicitation of preferences among alternatives leading to the construction of utility functions The ordering of alternatives assumes that transitivity is satisfied. If transitivity is violated, one cannot construct utility functions. On the other hand, when the alternatives are ordered through paired comparisons, which can violate transitivity, one can still construct a utility function. The following example illustrates this point. Assume that the ordinal preferences are replaced by 2¥2 reciprocal matrices as in (1). Transitivity can now be relaxed because a scale can be constructed from the set of paired comparisons representing the relative dominance of an alternative over the others. The resulting scale, $w(x)$, is given by the principal right eigenvector of the matrix of paired comparisons. This scale can now be used to construct a utility function as follows:

$$u(x) = \frac{w(x) - \min_x\{w(x)\}}{\max_x\{w(x)\} - \min_x\{w(x)\}}. \tag{2}$$

It can be easily shown (Vargas, 1986) that $u(x)$ is a true utility function, that is,

(1) if alternative A is preferred or indifferent to alternative B then u(A) ≥ u(B),

(2) if $L = \begin{pmatrix} p & 1-p \\ A & B \end{pmatrix}$ is a lottery, then $u(L)$ = pu(A) + (1-p)u(B).

Choice - when Irrelevant Alternatives Become Relevant

Choice theory is concerned with predicting the choices of individuals by making assumptions that may be mathematically appealing but are violated in practice. In choice theory we have another example where use of ordinal judgments leads to the violation of one of the assumptions known as the regularity principle. This principle is part of Luce's axiom [1959, p.6]:

Let T be a finite subset of U such that, for every S⊂T, P_S is defined.

(i) Independence from Irrelevant Alternatives:

If P(x,y) π 0,1 for all x,y ∈T, then for R ⊂S ⊂T ,

$$P_T(R) = P_S(R)P_T(S)\,;$$

(ii) Essentiality Principle

If P(x,y) = 0 for some x,y ∈T, then for every S ⊂T,

$$P_T(S) = P_{T-\{x\}}(S - \{x\})\,.$$

A consequence of the principle of independence from irrelevant alternatives is what is known as the principle of regularity (Farquhar and Pratkanis, 1987, p.7). It simply says that the probability of an alternative being chosen cannot be increased or decreased by adding (or deleting) alternatives to the choice set. An example where this principle is violated is that of individuals who are asked to choose between a product that sells for $300, and is consider too expensive, and another product that sells for $150 that is considered to be reasonably priced. The first product is less preferred than the second. If another product similar to the expensive one is introduced (sometimes only advertised as a future product but eventually never made) at the much higher price of $1,000, the $300 product becomes the more attractive and preferred one and there is a reversal in preference due to the presence of an irrelevant alternative. If instead of ordinal preferences we use cardinal preferences, the problem would be seen in a different light according to the degree of inconsistency and the strength of relative preference and the shift in preference would be easily explained thus justifying why rank should in fact reverse; the axiom notwithstanding -- simply an infeasible assumption no matter how attractive it may seem at first.

Pareto Optimality - Cardinality Always Yiels Definite Answers

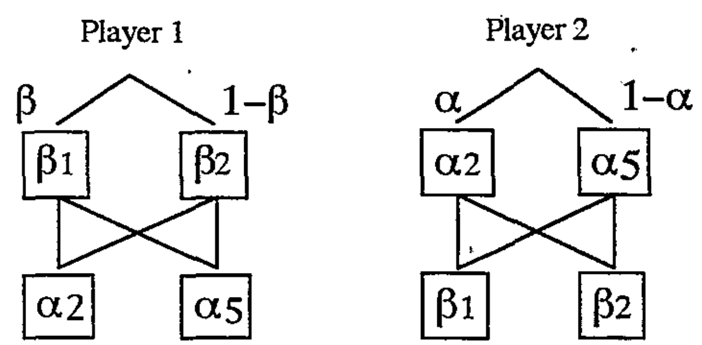
Pareto optimality is often used in conjuction with ordinal preferences as in game theory to study the ranking of alternatives. Here too problems arise. The Pareto optimal set is the set of all non dominated alternatives. In a 2-person game situation, the alternatives are pairs of strategies, (α_i, β_i), one strategy for each of the players. Consider for example the following matrix of relative payoffs from Saaty (1979):

$$\alpha_i \overset{\beta_j}{\begin{pmatrix} (.0190,0310) & (.0099,.0510) & (.0091,.0260) & (.0112,.0310) \\ (.0024,.1570) & (.0083,.0160) & (.0270,.0030) & (.0068,.1420) \\ (.2260,.0580) & (.1290,.0220) & (.1190,.0030) & (.2270,.1740) \\ (.0023,.0770) & (.0023,.0250) & (.0021,.0400) & (.0022,.0250) \\ (.0790,.0310) & (.1206,.0500) & (.0330,.0100) & (.0690,.0910) \end{pmatrix}}.$$

The strategy (α_3, β_4) provides the largest payoffs to both players. This is the Pareto dominant equilibrium. If this point is removed from the set of strategies, and hence also its row and column we get:

$$\alpha_i \overset{\beta_j}{\begin{pmatrix} (.0190,0310) & (.0099,.0510) & (.0091,.0260) & (\;,\;) \\ (.0024,.1570) & (.0083,.0160) & (.0270,.0030) & (\;,\;) \\ & & & \\ (.0023,.0770) & (.0023,.0250) & (.0021,.0400) & (\;,\;) \\ (.0790,.0310) & (.1206,.0500) & (.0330,.0100) & (\;,\;) \end{pmatrix}}.$$

Two Pareto optimal payoffs appear, (α_2, β_1) and (α_5, β_2). The decisions now can be represented with two hierarchies, one for each player:



where α and β are the likelihoods of player 1 and player 2, respectively, selecting the corresponding strategy. Normalizing the payoffs to unity for each of the strategies of the opponent and applying the principle of hierarchic composition we obtain the following composite priorities for the strategies of each player:

$$\begin{matrix} \alpha_2 \\ \alpha_5 \end{matrix} \overset{\underline{Player 1}}{\begin{pmatrix} .06+.17\beta \\ .94-.17\beta \end{pmatrix}} \text{ and } \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} \overset{\underline{Player 2}}{\begin{pmatrix} .38+.53\alpha \\ .62-.53\alpha \end{pmatrix}}.$$

Note that Player 1 should always select the α_5 strategy because it always has the largest priority for all values of β. Player 2 should select the strategy, β_2. if its priority is greater than .5 or $.62-.53\alpha > .5$. Thus, as long as Player 1 selects α_5 with a likelihood $1-\alpha > 1-((.62-.5)/.53) = 41/53$, β_2 seems to be the most preferred alternative. The point (α_5, β_2) is an equilibrium point for this game and the ambiguity of two Pareto solutions disappears with the use of cardinal preferences.

Outranking - No dominance, No decision

Outranking methods are based on ordinal preferences. The first of these methods (Electre I) was developed by Bernard Roy (1968). He introduced the concept of *an outranking relation* S as a binary relation defined on the set of alternatives A. Given two alternatives A_i and A_j, A_i outranks A_j, or A_iSA_j, if given all that it is known about the two alternatives, there are enough arguments to decide that A_i is at least as good as A_j. The goal of outranking methods is to find all alternatives that dominate other alternatives while they cannot be dominated by any other alternative. To find the best alternative the criteria weights are assumed to be measured on some scale, probably a ratio scale. Each criterion $C_j \in C$ is assigned a weight w_j, and every pair of alternatives A_i and A_j is assigned a concordance index $c(A_i,A_j)$ given by:

$$c(A_i,A_j) = \frac{1}{\sum_{k=1}^{n} w_k} \sum_{\{k:g_k(A_i)\geq g_k(A_j)\}} w_k$$

and a discordance index $d(A_i,A_j)$ given by:

$$d(A_i,A_j) = \begin{cases} 0 & if \quad g_k(A_i) \geq g_k(A_j) \quad for \quad all \quad k, \\ \frac{1}{\delta}\max_k\{g_k(A_j) - g_k(A_i)\}, & otherwise. \end{cases}$$

where $\delta = \max_{k,A_i,A_j}\{\{g_k(A_i) - g_k(A_j)\}$. Obviously, the discordance index is only valid if the operation of subtraction is well defined. Once the two indices are defined, an outranking relation S is defined by :

$$A_i \ S \ A_j \text{ if and only if } \begin{cases} c(A_i,A_j) \geq \hat{c}, \\ d(A_i,A_j) \leq \hat{d}, \end{cases}$$

where $\hat{c}$ and $\hat{d}$ are thresholds. A problem with this discordance index is that the criteria levels must be quantifiable. If that is not the case, then a discordance set D_j is defined for each criterion j with all the ordered pairs (x_j,y_j) such that if $g_j(A) = x_j$ and $g_j(B) = y_j$ then the outranking of B by A is denied. The outranking relation is now defined:

$$A_i \ S \ A_j \text{ if and only if } \begin{cases} c(A_i,A_j) \geq \hat{c}, \\ (g_j(A_i), g_j(A_j)) \notin D_j, \forall j. \end{cases}$$

Given the outranking relation it is now possible to find the set of alternatives $N \subseteq A$ for which:

$\forall B \in A\text{-}N, \exists A \in N$ such that ASB

$\forall A, B \in N$, A $\not S$ B.

The outranking relation determines the set of non-dominated alternatives. The alternatives in N form the kernel of the graph defined by the alternatives (vertices) and the outranking relation (edges). Thus, if alternative A_i outranks alternative A_j, then a directed arc exists from A_i to A_j: $A_i \rightarrow A_j$.

There are three other variations of this method depending on how the outranking relation is defined. The method most employed in applications requiring ranking of the alternatives rather than choice is based on an outranking relation in which the concordance and the discordance indices have two levels used to define a strong and a weak outranking relation. This method is known as Electre II (Roy and Bertier, 1973).

A group of researchers in the process of solving a problem developed software which can be used in a variety of forms to accomplish objectives such as (1) do research and obtain funds to buy the researchers time (RESEARCH), (2) develop a product and market it (MKTDEV), (3) capture some share of the market in the industry in question (MKTSHARE), and (4) make money (PROFIT). These objectives can be attained following different courses of action: (a) independent commercialization of the product

(INDCOMM), (b) form a joint venture with a company that has pursued similar projects in the past (JOINTVEN), (c) relinquish the right of the product to the institution where they are affiliated and collect royalties (NORIGHTS), and (d) obtain funding from independent sources and use them to do research and consulting with the tool developed (INDFUNDS). The matrices of paired comparisons and the corresponding priorities are given in Table 1.

The decision the AHP model suggests is to obtain independent funding and use it to do R&D (0.389). A close second alternative is to pursue a joint venture with a company that has done this type of work in the past (0.335).

The outranking method Electre I uses the outranking relation concept based on two indices, the concordance index and the discordance index. The former could be constructed using, for example, the priorities obtained in the AHP. For example, the concordance index of the alternatives a and b, C(a,b), is obtained by summing the weights of the criteria for which alternative a dominates alternative b. We have

$$C(a,b) = 0.199 + 0.084 = 0.283,$$
$$C(b,a) = 0.479 + 0.238 = 0.717.$$

Table 1. Pairwise comparisons and Priorities from the AHP model

Best Option	1	2	3	4	Priorities
1.RESEARCH	1	3	3	3	0.479
2.MKTDEV		1	1	3	0.199
3.MKTSHARE			1	5	0.238
4. PROFIT				1	0.084
					CR = 0.099
RESEARCH	a	b	c	d	Priorities
a	1	1/5	3	1/7	0.082
b		1	5	1/5	0.230
c			1	1/9	0.044
d				1	0.644
					CR = 0.111
MKTDEV	a	b	c	d	Priorities
a	1	3	7	5	0.574
b		1	5	2	0.239
c			1	1/3	0.056
d				1	0.131
					CR = 0.029
MKTSHARE	a	b	c	d	Priorities
a	1	1/5	1	1	0.112
b		1	7	7	0.666
c			1	1/3	0.080
d				1	0.141
					CR = 0.063
PROFIT	a	b	c	d	Priorities
a	1	3	2	2	0.470
b		1	5	1	0.225
c			1	1/5	0.060
d				1	0.244
					CR = 0.044

Actions	Composite Priorities
a.INDCOMM	0.220
b.JOINTVEN	0.335
c.NORIGHTS	0.056
d.INDFUNDS	0.389
	CR = 0.09

The resulting index is given in Table 2.

Table 2. The Concordance Index

C(x,y)	a	b	c	d
a	—	0.283	1	0.283
b	0.717	—	1	0.238
c	0	0	—	0
d	0.717	0.762	1	—

Table 3 summarizes the discordance index for all the pairs of alternatives.

Table 3. The Discordance Index

d(x,y)	a	b	c	d
a	—	0.558	0	0.738
b	0.923	—	0	0.875
c	0.863	0.977	—	1
d	0.937	0.690	0	—

These two indices are now used to construct the outranking relation. First, for an alternative to outrank another, we must select the thresholds $\hat{c}$ and $\hat{d}$ above and below which the concordance and the discordance indices, respectively, must fall. For example, if $\hat{c} = 0.25$ and $\hat{d} = .50$ then we obtain the graph given in Figure 1. Here the alternatives a, b and d outrank c but nothing can be said about whether or not one prefers one alternative over another.

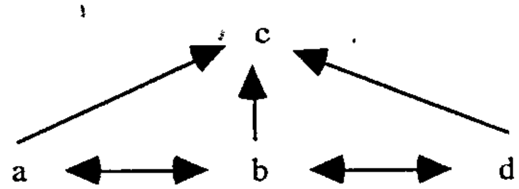


Figure 1

Making the discordance threshold $\hat{d} = 0.70$ we obtain the graph given in Figure 2. Here it is possible to conclude that alternative d dominates the others.

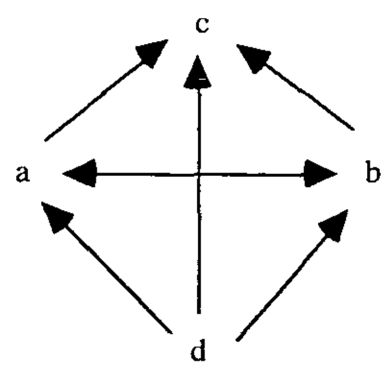


Figure 2

As one varies the concordance and discordance thresholds one gets different answers. One must justify the selection of the threshold which seems to be obscure in Electre. In addition, in Figure 1 Electre gives a different result than the AHP, which should one believe?

Conclusions

Using cardinal preferences simplifies decision making situations that have been obscured in the past through the use of ordinal preferences. Any situation that leads to a non decision, to a paradox or to an inappropriate use of scales through ordinal preferences should be reexamined through cardinal preferences. From the foregoing discussion, it is clear that the use of ordinals does not layout a successful path to making decisions that are consistent with human abilities and human behavior. Feelings, emotions and thoughts are not inherently ordinal, and our theories of decision making should take these factors into account.

References

Farquhar, P.H. and A.R. Pratkanis, "Phantom Choices: The Effects of Unavailable Alternatives on Decision Making," Technical Report 87-2, Decision Research Program. GSIA, Carnegie-Mellon University, 1987.

Luce, D.R., *Individual Choice Behavior*, New York: Wiley, 1959

Peniwati, K., Ph.D. Dissertation. In progress. KGSB, University of Pittsburgh, 1996.

Roy, B., "Classement et Choix en Présence de Points de Vue Multiples (la Méthode Electre)," *Revue Française d'Informatique et de Recherche Opérationalle* 8 (1968) 57-75.

Roy, B. and P. Bertier, La Méthode ELECTRE II, une Application au Média-planning," In *OR* 72, M. Ross (ed), North-Holland, 291-302, 1973.

Saaty, T.L., "The U.S.-OPEC Energy-Conflict: The Payoff Matrix by the Analytic Hierarchy Process," *Int'l Journal of Game Theory* 8, 4 (1979) 225-234.

Vargas, L.G., "Utility Theory and Reciprocal Pairwise Comparisons: The Eigenvector Method," *Journal of Socio-Economic Planning Sciences* 20, 6 (1986) 387-391.