AHP EIGENVECTOR
VIA DYNAMIC PROCESS OF PAIRWISE COMPARISON AND AVERAGING

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ABSTRACT

The essentiality of using the eigenvector of a pairwise comparison matrix for creating the associated priority weight vector is explained by way of introducing a dynamic process which is related to the value evolution and the value establishment among compared items. The dynamic process describes how the priority weight of each item is determined in time through evaluating pairwise comparison values and averaging them. It is shown that when the arithmetic mean is used for averaging the pairwise comparison values, the dynamic process results in Saaty's Eigenvector Method for the case of complete-information pairwise matrix and that the dynamic process results in Harker's Eigenvector Method for the case of incomplete-information pairwise matrix. It is also shown that the convergency of the dynamic process is dependent on the Consistency Index value of the associated comparison matrix.

Keywords: eigenvector in AHP, dynamics of AHP, convergence properties of evaluation process, Saaty's Eigenvector Method, Harker's Eigenvector Method, Consistency Index

1. Introduction

The essentiality, or even the inevitability, of using the eigenvector of a pairwise comparison matrix for creating the associated priority weight vector is explained by way of introducing a dynamic process which is related to the value evolution and the value establishment among compared items.
In Chapter 2, the dynamic process, or the dynamics, of pairwise comparison evaluation and averaging is explained by two examples, a complete-information case and an incomplete-information case, and then its general form is presented for each case. In Chapter 3, the stability of the dynamic process, or the convergence properties of the dynamic process, is studied in relation to the Consistency Index value of the associated comparison matrix.

2. Dynamic process of pairwise comparison and arithmetic mean averaging

The dynamic process of pairwise comparison and arithmetic mean averaging is explained by two 4-item examples, a complete-information case in Section 2.1 and an incomplete-information case in Section 2.2, and then its general form is presented for each case in Sections 2.3 and 2.4.

2.1 Example of 4-item complete-information case

First, consider an example of dynamic process for comparing among 4 items with a complete-information pairwise comparison matrix $\boldsymbol{A}$ given by (1) and the arithmetic mean averaging (2).

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \tag{1}$$

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$$w_k(\text{new}) \leftarrow \frac{1}{4}\left(a_{k1}w_1(\text{old}) + a_{k2}w_2(\text{old}) + a_{k3}w_3(\text{old}) + a_{k4}w_4(\text{old})\right) \quad (k = 1,2,3,4) \tag{2}$$

New weight of item k, $w_k(\text{new})$, is determined by averaging four pairwise comparison evaluated values w.r.t. item k, $a_{k1}w_1(\text{old})$, $a_{k2}w_2(\text{old})$, $a_{k3}w_3(\text{old})$, and $a_{k4}w_4(\text{old})$, using the arithmetic mean.
By introducing the notation, time instance t, this dynamic process can be written by (3), where N = 4, and its continuous-time form is given by (4), where $\boldsymbol{I}$ is the unit matrix.

$$\boldsymbol{w}(t+1) = \frac{1}{N}\boldsymbol{A}\boldsymbol{w}(t) \tag{3}$$

$$\frac{d\boldsymbol{w}(t)}{dt} = \left(\frac{1}{N}\boldsymbol{A} - \boldsymbol{I}\right)\boldsymbol{w}(t) \tag{4}$$

2.2 Example of 4-item incomplete-information case

Next, consider an example of the dynamic process for comparing among 4 items with an incomplete–information pairwise comparison matrix $\boldsymbol{A}$ given by (5), where ( ) denotes a missing element, and the arithmetic mean averaging (6).

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & (\ ) & a_{14} \\ a_{21} & a_{22} & a_{23} & (\ ) \\ (\ ) & a_{32} & a_{33} & a_{34} \\ a_{41} & (\ ) & a_{43} & a_{44} \end{pmatrix} \tag{5}$$

$$\begin{aligned} w_1(\text{new}) &\leftarrow \frac{1}{3}\left(a_{11}w_1(\text{old}) + a_{12}w_2(\text{old}) + a_{14}w_4(\text{old})\right) \\ w_2(\text{new}) &\leftarrow \frac{1}{3}\left(a_{21}w_1(\text{old}) + a_{22}w_2(\text{old}) + a_{23}w_3(\text{old})\right) \\ w_3(\text{new}) &\leftarrow \frac{1}{3}\left(a_{32}w_2(\text{old}) + a_{33}w_3(\text{old}) + a_{34}w_4(\text{old})\right) \\ w_4(\text{new}) &\leftarrow \frac{1}{3}\left(a_{41}w_1(\text{old}) + a_{43}w_3(\text{old}) + a_{44}w_4(\text{old})\right) \end{aligned} \tag{6}$$

Multiplying 3 to the both sides of each updating formula of (6) and then adding w_k to the both sides of each updating formula, (7) is obtained.

$$\begin{aligned} 4w_1(\text{new}) &\leftarrow (a_{11}+1)w_1(\text{old}) + a_{12}w_2(\text{old}) + 0w_3(\text{old}) + a_{14}w_4(\text{old}) \\ 4w_2(\text{new}) &\leftarrow a_{21}w_1(\text{old}) + (a_{22}+1)w_2(\text{old}) + a_{23}w_3(\text{old}) + 0w_4(\text{old}) \\ 4w_3(\text{new}) &\leftarrow 0w_1(\text{old}) + a_{32}w_2(\text{old}) + (a_{33}+1)w_3(\text{old}) + a_{34}w_4(\text{old}) \\ 4w_4(\text{new}) &\leftarrow a_{41}w_1(\text{old}) + 0w_2(\text{old}) + a_{43}w_3(\text{old}) + (a_{44}+1)w_4(\text{old}) \end{aligned} \tag{7}$$

Then, its discrete-time form is given by (8), where $\boldsymbol{A}^*$ is given by (9) and N = 4.

$$\boldsymbol{w}(t+1) = \frac{1}{N}\boldsymbol{A}^*\boldsymbol{w}(t) \tag{8}$$

$$\boldsymbol{A}^* = \begin{pmatrix} a_{11}+1 & a_{12} & 0 & a_{14} \\ a_{21} & a_{22}+1 & a_{23} & 0 \\ 0 & a_{32} & a_{33}+1 & a_{34} \\ a_{41} & 0 & a_{43} & a_{44}+1 \end{pmatrix} \tag{9}$$

2.3 General form for the case of complete-information and arithmetic mean averaging

As shown from the example of Section 2.1, the general form of the dynamic process of pairwise comparison and arithmetic mean averaging for the case of complete-information and arithmetic mean averaging is given by (10).

$$\boldsymbol{w}_0(t+1) = \frac{1}{N}\boldsymbol{A}\boldsymbol{w}_0(t) \tag{10}$$

Here, $\boldsymbol{w}_0(t)$: weight vector at time t (defined by (10)),
$\boldsymbol{A}$: complete-information comparison matrix,
N : number of items.

The eigenvalue problem formulation of (10) is given by (11).

$$\frac{1}{N}\boldsymbol{A}\boldsymbol{w}_0 = \lambda_0 \boldsymbol{w}_0 \tag{11}$$

Here, $\boldsymbol{w}_0$: right eigenvector for matrix $\frac{1}{N}\boldsymbol{A}$,
λ_0 : right eigenvalue for matrix $\frac{1}{N}\boldsymbol{A}$.

Saaty's Eigenvector Method (Saaty, 1980), or the Power Method for obtaining the eigenvector and the eigenvalue, can be written by (12).

$$\boldsymbol{w}_s(t+1) = \boldsymbol{A}\boldsymbol{w}_s(t) \tag{12}$$

Here, $\boldsymbol{w}_s(t)$: weight vector at time t (defined by (12)).
The eigenvalue problem formulation of (12) is given by (13).

$$\boldsymbol{A}\boldsymbol{w}_s = \lambda_s \boldsymbol{w}_s \tag{13}$$

Here, $\boldsymbol{w}_s$: right eigenvector for matrix $\boldsymbol{A}$,
λ_s : right eigenvalue for matrix $\boldsymbol{A}$.

[Comment 1]

Since $\lambda_0 = \lambda_s / N$ and $\boldsymbol{w}_0 = \boldsymbol{w}_s$, Saaty's Eigenvector Method and the proposed dynamic process of pairwise comparison and arithmetic mean averaging are regarded the same for the case of a complete-information pairwise comparison matrix. □

2.4 General form for the case of incomplete-information and arithmetic mean averaging

As shown from the example of Section 2.2, the general form of the dynamic process for the case of incomplete-information and arithmetic mean averaging is given by (14).

$$w_k(\text{new}) \leftarrow \frac{1}{|S_k|}\sum_{j \in S_k} a_{kj} w_j(\text{old}) \quad (k = 1,2,\ldots,N) \tag{14}$$

Here, S_k: the set of items which are (directly) compared with item k.

Multiplying $|S_k|$ to the both sides of (14), (15) is obtained.

$$|S_k| w_k(\text{new}) \leftarrow \sum_{j \in S_k} a_{kj} w_j(\text{old}) \quad (k = 1,2,\ldots,N) \tag{15}$$

Adding $(N - |S_k|)w_k$ to the both sides of (15), (16) is obtained.

$$N w_k(\text{new}) \leftarrow \sum_{j \in S_k} a_{kj} w_j(\text{old}) + (N - |S_k|) w_k(\text{old}) \quad (k = 1,2,\ldots,N) \tag{16}$$

Then, the discrete-time form of (16) is given by (17).

$$w_k(t+1) = \frac{1}{N}\left\{\sum_{j\in S_k} a_{kj} w_j(t) + \left(N - |S_k|\right) w_k(t)\right\} \quad (k = 1, 2, \ldots, N) \tag{17}$$

Noting that $N - |S_k|$ is the number of missing elements in the kth row of an $N \times N$ incomplete-information pairwise comparison matrix $\boldsymbol{A} = \{a_{kj}\}$, the matrix-vector form of (17) is given by (18).

$$\boldsymbol{w}_1(t+1) = \frac{1}{N}\boldsymbol{A}^*\boldsymbol{w}_1(t) \tag{18}$$

Here, $\boldsymbol{w}_1(t)$: weight vector at time t (defined by (18)),

$\boldsymbol{A}^* = \{a_{kj}{}^*\}$: Harker's modified matrix, defined as below.

if $k \neq j$ and $a_{kj} > 0$, then $a_{kj}{}^* = a_{kj}$.

if $a_{kj} = (\ )$, or missing, then $a_{kj}{}^* = 0$.

if $k = j$, then $a_{kk}{}^* = a_{kk} + N - |S_k|$.

Using Harker's modified comparison matrix $\boldsymbol{A}^*$, Harker's Eigenvector Method (Harker, 1987) can be written by (19).

$$\boldsymbol{w}_{\mathrm{H}}(t+1) = \boldsymbol{A}^*\boldsymbol{w}_{\mathrm{H}}(t) \tag{19}$$

Here, $\boldsymbol{w}_{\mathrm{H}}(t)$: weight vector at time t (defined by (19)).

The eigenvalue problem formulations of (18) and (19) are given by (20) and (21), respectively.

$$\frac{1}{N}\boldsymbol{A}^*\boldsymbol{w}_1 = \lambda_1\boldsymbol{w}_1 \tag{20}$$

$$\boldsymbol{A}^*\boldsymbol{w}_{\mathrm{H}} = \lambda_{\mathrm{H}}\boldsymbol{w}_1 \tag{21}$$

Here, $\boldsymbol{w}_1$: right eigenvector for matrix $\frac{1}{N}\boldsymbol{A}^*$,

λ_1 : right eigenvalue for matrix $\frac{1}{N}\boldsymbol{A}^*$,

$\boldsymbol{w}_{\mathrm{H}}$: right eigenvector for matrix $\boldsymbol{A}^*$,

λ_{H} : right eigenvalue for matrix $\boldsymbol{A}^*$.

[Comment 2]

Since $\lambda_1 = \lambda_{\mathrm{H}}/N$ and $\boldsymbol{w}_1 = \boldsymbol{w}_{\mathrm{H}}$, Harker's Eigenvector method and the proposed dynamic process of pairwise comparison and arithmetic mean averaging can be regarded the same for the case of an incomplete-information pairwise comparison matrix. □

[Comment 3]

Another weight vector updating formula for the incomplete-information case can be given by (22), directly from the discrete-time form of (14). This is considered as an original extension version of Saaty's Eigenvector Method to the incomplete-information case from the viewpoint of proposed dynamic process of pairwise comparison and averaging.

$$\boldsymbol{w}_2(t+1) = \boldsymbol{B}\boldsymbol{w}_2(t) \tag{22}$$

Here, $\boldsymbol{w}_2(t)$: weight vector at time t (defined by(22)),

$\boldsymbol{B} = \{b_{kj}\}$: modified comparison matrix from $\boldsymbol{A} = \{a_{kj}\}$, defined as below.

if $a_{kj} \neq (\ )$, then $b_{kj} = a_{kj}/|S_k|$.

if $a_{kj} = (\ )$, or missing, then $b_{kj} = 0$. □

3. Stability of the dynamic process

The stability, or the convergence properties, of the dynamic process of pairwise comparison and

arithmetic mean averaging, described by (23) and (24), will be studied in this chapter.
[Discrete-time form]

$$\boldsymbol{x}(t+1)=\frac{1}{N}\boldsymbol{A}\boldsymbol{x}(t) \tag{23}$$

[Continuous-time form]

$$\frac{d\boldsymbol{x}(t)}{dt}=\left(\frac{1}{N}\boldsymbol{A}-\boldsymbol{I}\right)\boldsymbol{x}(t) \tag{24}$$

Let λ_p and $\boldsymbol{x}_p$ be the right principal eigenvalue and eigenvector for matrix $\boldsymbol{A}$, respectively. Then, the right principal eigenvalue and eigenvector for matrix $\frac{1}{N}\boldsymbol{A}$ are given by $\frac{\lambda_p}{N}$ and $\boldsymbol{x}_p$, respectively, and those for matrix $\frac{1}{N}\boldsymbol{A}-\boldsymbol{I}$ are given by $\frac{\lambda_p}{N}-1$ and $\boldsymbol{x}_p$, respectively. The Saaty's Consistency Index value CI for matrix $\boldsymbol{A}$ and eigen weight vector $\boldsymbol{w}_s$, is defined by (25), using λ_p, the right principal eigenvalue of $\boldsymbol{A}$.

$$\mathrm{CI}=\frac{\lambda_p-N}{N-1} \tag{25}$$

Therefore, the eigenvalue $\lambda_{p1}=\frac{\lambda_p}{N}$ for $\frac{1}{N}\boldsymbol{A}$ and the eigenvalue $\lambda_{p2}=\frac{\lambda_p}{N}-1$ for $\frac{1}{N}\boldsymbol{A}-\boldsymbol{I}$ are expressed by (26) and (27) using CI.

$$\lambda_{p1}=\frac{\lambda_p}{N}=1+\frac{N-1}{N}\mathrm{CI} \tag{26}$$

$$\lambda_{p2}=\frac{\lambda_p}{N}-1=\frac{N-1}{N}\mathrm{CI} \tag{27}$$

[Comment 4]
Let CI be defined by (25) with λ_p as the right principal eigenvalue of pairwise comparison matrix $\boldsymbol{A}$ for the complete-information case, and of Harker's modified matrix $\boldsymbol{A}^*$ for the incomplete-information case. If CI = 0, then the dynamic process of pairwise comparison and arithmetic mean averaging, given by (10) or (18), converges to a stationary eigen weight vector. □

[Comment 5]
If CI > 0, then the dynamic process itself, given by (10) or (18), does not converge to a weight vector, but diverges. The weight multiplying factor λ_p increases with the CI value (see (25)). □

[Comment 6]
When CI > 0, although the magnitude of the weight vector $\boldsymbol{w}_0(t)$ in (10) or $\boldsymbol{w}_1(t)$ in (18) increases with time *t*, the elementwise ratio of the weight vector $\boldsymbol{w}_0(t)$ or $\boldsymbol{w}_1(t)$, converges. That is, if the weight vector $\boldsymbol{w}_0(t)$ or $\boldsymbol{w}_1(t)$ is normalized, such that the sum of each element of the weight vector $\boldsymbol{w}_0(t)$ or $\boldsymbol{w}_1(t)$ is unity, the normalized dynamic process of $\boldsymbol{w}_0(t)$ in (10) or $\boldsymbol{w}_1(t)$ in (18) converges to a (normalized) weight vector. □

The Saaty's CI value CI is defined also by (28), using a_{kj} and w, instead of using λ_p.

$$\mathrm{CI}=\frac{1}{N(N-1)}\left(\sum_{k\neq j}a_{kj}\left(\frac{w_j}{w_k}\right)\right)-1 \tag{28}$$

Similarly, another Consistency Index value CJ can be defined by (29), where CI and CJ are related by (30).

$$\text{CJ} = \frac{1}{N^2}\left(\sum a_{kj}\left(\frac{w_j}{w_k}\right)\right) - 1 \tag{29}$$

$$\text{CJ} = \frac{N-1}{N}\text{CI} \tag{30}$$

[Comment 7]

The stability of discrete-time dynamic system (23) is characterized by the principal eigenvalue λ_{p1} of (26) and the stability of continuous-time dynamic system (24) is characterized by the principal eigenvalue λ_{p2} of (27). The eigenvalues λ_{p1} and λ_{p2} are expressed more simply by using CJ of (30) instead by using CI of (28) as shown by (31) and (32).

$$\lambda_{p1} = 1 + \text{CJ} \tag{31}$$

$$\lambda_{p2} = \text{CJ} \tag{32}$$

□

4. Conclusions

The dynamic process of pairwise comparison and averaging, which is supposed to simulate the dynamics of the priority weight determination process in our human mind from a set of pairwise comparison judgments, is presented and the stability of the dynamic process, or the convergence properties of the dynamic process, is studied in relation to the Consistency Index value. This study result suggests, that, roughly speaking, the dynamic process of creating priority weight vector in our human mind is stabilized, or converges, sooner or later, when the measured set of pairwise comparison judgments is consistent, but the process is unstable, or diverges, when the measured set of pairwise comparison judgments is inconsistent. The former corresponds to the judgment convergence in our mind and the latter corresponds to the judgment divergence in our mind. Even when the judgment, or the priority weight vector, diverges, its diverging direction suggests a judgment result, which is the EIGENVECTOR.

In this paper, the linear and deterministic version of the dynamic process of pairwise comparison and arithmetic mean averaging, in connection with the time-invariant pairwise comparison matrix $\boldsymbol{A}$, is studied. Since the pairwise comparison judgment consists of a part of the entire AHP decision making process, followings are considered to be some of the future research subjects.

1. Extension of the proposed dynamic process to the entire AHP decision making and proposal of DAHP, Dynamic Analytic Hierarchy Process.
2. Application of various averaging functions and summary statics other than the arithmetic mean to the averaging of pairwise comparison values, such as the harmonic mean, the geometric mean (including various weighted means), the median, the maximum, the 2nd maximum, the minimum, etc (Kanari and Shinohara, 2011).
3. Studies on the nonlinear and the stochastic versions of the dynamic process and the time-variant version of pairwise comparison matrix $\boldsymbol{A}(t)$.

REFERENCES

Saaty, T.L. (1980). *The Analytic Hierarchy Process*, McGraw-Hill, New York.

Harker, P.T. (1987). Alternative Modes of Questioning in the Analytic Hierarchy Process, *Math. Modelling, 9*, 353-360.

Kanari, K., & Shinohara, M. (2011). AHP Eigenvector Interpreted from the Model of Averaging Pairwise Evaluation Values and Eigenvector Concept in Path Algebra. *Proceedings of the 43rd Annual Meeting of College of Industrial Technology, Mathematical Information Engineering Division, Nihon University*, 103-106.