ISAHP 1999, Kobe, Japan, August 12-14, 1999

Analyzing AHP-matrices by Robust Regression*

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Abstract In the analytic hierarchy process (AHP) the decision maker makes comparisons between pairs of entities of interest. The comparisons can be thought of being influenced by random errors, and then the values of the ratios of the weights of the entities are values of random variables. Sometimes the ratio may be exceptionally different from the corresponding consistent value. Then the statement is called an outlier. In this paper we study the influence of the outlier on the estimates of the weights calculated by the eigenvector method and by the regression technique. It can be seen that outliers can have a significant influence on the weight estimates given by the eigenvector method and the logarithmic least squares regression. Here, we present the method of the logarithmic robust regression, which is robust in the presence of the outliers. We show by illustrative simulations how the solution of the logarithmic robust regression remains stable under random occurrences of outliers.

Key Words Analytical Hierarchy Process, eigenvector, regression, robust regression.

1 INTRODUCTION

The standard method to calculate the values for the weights from an AHP-matrix is to take the eigenvector corresponding to the largest eigenvalue of the matrix, and then to standardize the sum of the components equal to one (Saaty, 1977, 1980). A drawback of this method is that there is no practical statistical theory behind it. A statistical approach is needed if one thinks that the ratios of the relative importance of entities contain random fluctuations. If one looks at real AHP-matrices from real experiments, the feeling of randomness comes to mind. There are also studies showing that in human decision making inconsistencies can be expected, see for instance Fischoff et al. (1980). In this paper we do not, however, consider the effects of the comparison scale on the inconsistencies (Pöyhönen et al. 1997, Salo and Hämäläinen 1997).

A non-statistical approach to process imprecise or nonconsistent judgments is to embed them into intervals. This idea was first proposed by Arbel (1989), and futher developed by Zahir (1991), Arbel and Vargas (1992), Salo (1993), and Salo and Hämäläinen (1995). Arbel and Vargas, and Zahir have an optimization approach, while Salo and Hämäläinen process the judgements as constraint intervals. Saaty and Vargas (1987) study the distribution on the judgment interval, and they have done a sampling experiment to study the impact of imprecise pairwise judgments on the weight estimates.

*Submitted to the Fifth International Symposium on the Analytic Hierarchy Process (ISAHP'99)

Crawford and Williams (1985), and Alho et al. (1996) propose the use of a regression analysis for analyzing AHP-matrices. In this approach the ratios given by the decision maker are thought as values of random variables. This method often gives very similar weights as the eigenvector method, and it has the advantage that it is supported by a well-known statistical theory.

In an ideal AHP-matrix the pairwise comparisons are fully consistent. If an expert makes the pairwise comparisons one by one, there is random variation in the values of ratios, and sometimes an entity deviates considerable from the consistent value of the corresponding element. The following 4×4-matrix is an example from a real case study (Kangas et al. 1992).

$$\begin{bmatrix} 1 & 3 & 7 & 6 \\ 1/3 & 1 & 5 & 3 \\ 1/7 & 1/5 & 1 & 2 \\ 1/6 & 1/3 & 1/2 & 1 \end{bmatrix}$$

The consistency ratio CR of this AHP-matrix is CR=0.069. The main reason for the inconsistency is the entry (2,3) with the value of 5. The value 7/3 for the entry (2,3) should be consistent with the comparisons on the first row. In the regression analysis a deviating value is called an outlier, and here we follow the same term. The value 5 of the entry (2,3) has a strong influence on the values of weights given by the standard calculating methods. The eigenvector solution gives the weights 0.5762, 0.2648, 0.0874, 0.0716, and the logarithmic least squares regression gives the corresponding values 0.5834, 0.2604, 0.0851, 0.0711. If we put the value of 7/3 for the entry (2,3), and the value of 3/7 for the entry (3,2), the weights are the following: the eigenvector solution 0.6011, 0.2185, 0.1066, 0.0738, and the logarithmic least squares regression solution 0.5998, 0.2213, 0.1059, 0.0731. It can be seen that the inconsistency in the entry (2,3) has a noticeable effect on the weights.

The regression approach gives a possibility to use a technique which is robust to outliers. The logarithmic robust regression applied to the matrix above gives the weights 0.6009, 0.2233, 0.0952, 0.0805 which are near the values given by both the eigenvector method and the logarithmic least squares regression applied to the matrix where the entry (2,3) has been changed consistent with the comparisons on the first row.

Our aim is to highlight the use of the robust regression in the analysis of AHP-matrices. In Chapter 2 we present the regression approach and the method of the robust regression. In Chapter 3 we demonstrate with thorough simulations the differences of the solutions given by the eigenvector method and the robust regression.

2 THE METHOD OF ROBUST REGRESSION

The regression approach

Let us take an AHP-matrix of size $m \times m$ with the entries r_{ij}, $i, j = 1, ..., m$. So, r_{ij} is the relative value of attribute i compared to attribute j as perceived by the decision maker (DM). The entries r_{ij} make the pairwise comparisons data with the reciprocal relation $r_{ji} = 1/r_{ij}$ for $i, j = 1, ..., m$. We first show how data can be put into a regression form.

Let $w_1, ..., w_m$ be the true weights of the attributes with the condition $w_1 + \cdots + w_m = 1$, and let $v_1, ..., v_m$ be values of the attributes so that the normalized weights $w_1, ..., w_m$ can be calculated from $v_1, ..., v_m$ by normalization the sum equal one. Now, the observation r_{ij} is an observation from the ratio v_i/v_j, and the logarithm $\log(r_{ij})$ is an observation from $\log(v_i/v_j)$=$\log(v_i)$-$\log(v_j)$. For the total of $m(m-1)/2$ comparisons we can write the equations

$$\log(r_{ij}) = \log(v_i) - \log(v_j) + \epsilon_{ij}, = \beta_i - \beta_j + \epsilon_{ij} \ \ i = 2, ..., m, j = 1, ..., m-1, \qquad (1)$$

where ϵ_{ij} are random variables with expectation E[ϵ_{ij}]=0, and β_i=log(v_i), $i = 1, ..., m$.

The parameters β_i are solved from equations (1) with the assumption that the random variables ϵ_{ij} are independent and normally distributed with expectation E[ϵ_{ij}]=0 and with a common variance Var[ϵ_{ij}] $= \sigma^2$. That means the random variables $\log(r_{ij})$ are distributed as N($\beta_i - \beta_j, \sigma^2$). By normalization β_m=0 (v_m=1) one comes to the usual solution of the least squares. The solution can be computed by using a standard regression program.

The estimates of the weights $w_1, ..., w_m$ are calculated as follows:

$$\hat{w}_i = \frac{\exp(\hat{\beta}_i)}{\exp(\hat{\beta}_1) + \cdots + \exp(\hat{\beta}_m)}, \ \ i = 1, ..., m, \qquad (2)$$

where $\hat{\beta}_1, ..., \hat{\beta}_m$ are the estimates of $\beta_1, ..., \beta_m$ given by the regression analysis with the normalization $\hat{\beta}_m = 0$. The variances of the estimates $\hat{w}_i$ can be calculated from the variances of $\hat{\beta}_i$:s by using the delta method like Alho and Kangas (1997).

The robust regression approach

The theory of the robust regression can be found in standard statistical texts, see for instance Montgomery and Peck (1992).

We take a regression model in general terms

$$\mathbf{y} = \mathbf{X}\beta + \epsilon, \qquad (3)$$

where $\mathbf{y} = (y_1, ..., y_n)^T$ is the observation vector of the response Y in n trials, $\mathbf{X}$ is the design matrix of size $n \times m$ of the m independent regressors $X_1, ..., X_m$, $\beta = (\beta_1, ..., \beta_m)^T$ is the parameter vector, and $\epsilon = (\epsilon_1, ..., \epsilon_n)^T$ is a random vector of independent random errors ϵ_i, residuals with expectation zero. Let $\mathbf{x_i} = (x_{i1}, ..., x_{im})$ be the ith row of the matrix $\mathbf{X}$.

The general approach to estimation of the parameters β is to minimize a function d of the residuals,

$$\min_\beta \sum_{i=1}^{n} d(\epsilon_i) = \min_\beta \sum_{i=1}^{n} d(y_i - \mathbf{x_i}\beta). \qquad (4)$$

An estimate of this type is often called an M-estimate, where M stands for maximum likelihood.

There are lot of proposals for the function d. If we take $d(z) = z^2/2$ with a standardized variable z, the regression is the usual least squares regression. The function $d(z) = |z|$ gives the so called least absolute deviation regression. We use in this paper Andrew's wave function $d(z) = a[1 - \cos(z/a)]$ with the constant $a = 2.1$. We have chosen Andrew's method because of its popularity in the regression analysis. See Montgomery and Peck (1992).

3. DEMONSTRATIONS ON ROBUST REGRESSION

Let us look at the following AHP-matrix of size 4×4:

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 1/5 & 1 & 7/5 & 9/5 \\ 1/7 & 5/7 & 1 & 9/7 \\ 1/9 & 5/9 & 7/9 & 1 \end{bmatrix} \tag{5}$$

The consistency ratio of this AHP-matrix is CR=0. All the methods studied here, the eigenvector method, the logarithmic least squares regression, and the logarithmic robust regression give the same weights: 0.6878, 0.1376, 0.0983, 0.0764. We call these the correct weights.

Let us suppose that the judgment in comparison of the entities number three and four is not the consistent relation $9/7$ but it is $7/9$. So, the entry (3,4) is $7/9$ and the entry (4,3) is $9/7$ in the comparison matrix. This changes the value of CR to 0.012. The eigenvector method gives the weights 0.6878, 0.1376, 0.0872, 0.0874, the logarithmic least squares regression gives the values 0.6887, 0.1377, 0.0868, 0.0868, and the result of the logarithmic robust regression is 0.6878, 0.1376, 0.0983, 0.0764. One can see that the values of the two last weights have been changed both in the eigenvector and in the logarithmic least squares solution, but the logarithmic robust regression gives the correct solution.

If we change the entry (3,4) more, for instance if we give the value $1/2$, and the value 2 for the entry (4,3), then the solution of the eigenvector method and the logarithmic least squares regression change more considerable than above, but the solution of the logarithmic robust regression remains still the same. In this case CR=0.042, and the weights given by the eigenvector method are 0.6847, 0.1369, 0.0788, 0.0995, and by the logarithmic least squares regression 0.6880, 0.1376, 0.0776, 0.0968.

The example shows that the eigenvector method and the logarithmic least squares regression are sensitive to outliers which make the AHP-matrix inconsistency, but the solution given by the logarithmic robust regression remains stable.

A simulation study

In this example we take the matrix $\mathbf{A}$ given in (5) which is consistent and has the weights 0.6878, 0.1376, 0.0983, 0.0764. Then we generate from matrix $\mathbf{A}$ a new matrix where there are random errors in the judgments. First, we generate a normal distributed error for each judgment in the upper triangle by using the value zero as the mean and taking the standard deviation which is 10% from the consistent value. One can see by simulation that the coefficient of variation of 10% produces a significant variation of the estimates of the weights. Secondly, we take randomly one element of the upper triangle and give it the value 1. This is an outlier made randomly. Taking an element from the upper triangle means picking up randomly one of the possible comparisons of the entities. In some cases the value 1 is a strong outlier, in some cases it is a weak outlier only. Then we calculate the values for the weights both by using the eigenvector method and the logarithmic robust regression.

The above procedure has been repeated 200 times, and the descriptive statistics of the weights have been made. The descriptive statistics can be seen in table 1.

We have thought in our simulations that the ratios r_{ij} are continuous random variables with expectations given in matrix (5). We think that this will demonstrate the influence of the randomness.

Table 1 The means and the standard deviations of of the weights from 200 simulations. Eigenv. = the weights calculated by the eigenvector method. Rob.reg. = the weights calculated by the robust regression. * = the mean differs significantly from the correct weight at the 5% significance level.

Weight	Mean Eigenv.	Stdev Eigenv.	Mean Rob. reg.	Stdev Rob.reg.	Correct weight
w_1	0.6132*	0.0806	0.6854	0.0184	0.6878
w_2	0.1632*	0.0485	0.1366	0.0114	0.1376
w_3	0.1227*	0.0421	0.0987	0.0085	0.0983
w_4	0.1009*	0.0386	0.0793*	0.0076	0.0764

In table 1, the differences between the means given by the eigenvector method and the correct weights all are statistically highly significant. This tells that the eigenvector method does not give unbiased weights if there are random variations and outliers in the judgements. From the means got by the robust regression only the smallest one differs significantly from the correct weight, and the difference is very small. We do not see any considerable bias in the results of the robust regression.

There is also a noticeable difference between the standard deviations of the weights given by the two methods. The standard deviations of the weights calculated by the eigenvector method are much higher than the standard deviations when the robust regression has been used. The large standard deviations of the weights trough the eigenvector method indicate that the random variation of the judgments has a strong influence on the eigenvector solution. The small standard deviations of the weights given by the robust regression indicate the robustness of the method.

4 CONCLUSIONS

The ratios of the weights given by a decision maker can be thought as random variables with definite expectations and variances. Sometimes, it may happen that the decision maker gives a ratio which deviates exceptionally far from the expectation. This kind of an outlier has powerful influence on the values of the estimated weights given by the standard eigenvector method and the least squares regression method. The robust regression technique is a method which is not sensitive to outliers.

We have compared the use of the robust regression with the eigenvector method in analyzing AHP-matrices. If one generates outliers randomly in the AHP-matrix the estimates of the weights calculated by the eigenvector method vary considerably, but the estimates given by the robust regression remain stable. The robust regression gives the solution connected with the consistent part of the AHP-matrix. This is due to the weighting method of the robust regression. In this method large residuals are weighted slightly only.

In order to minimize the influence of outliers it is worthwhile to use the robust regression in addition to the eigenvector technique. If the eigenvector method gives the solution consistent with the solution of the robust regression, then there are no problems. But if there are significant differences between some weights calculated by different methods, then the AHP-matrix is inconsistent. For instance, if the rankings of the weights by different methods are not equal, then the solution of the eigenvector method is justified to a more detailed evaluation.

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