A GRAPHICAL APPROACH TO MODELING OF COMPLEX SYSTEMS IN AHP

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ABSTRACT

This paper is dealing with the structural characteristics of the complex system modeling in AHP by the approach of graph theory.

A (0-1)-matrix is firstly introduced to express the structure of a general system, either a simple hierarchy or a network with feedback, and this is illustrated by a connected graph. Some graphical expressions of the structural characteristics such as reducibility, primitivity, periodicity and their impact upon the convergence of the structure matrix are discussed based on the theory of Boolean matrix. After that, according to the relationship between the structure of the system and that of the digraph, a strict and systematical classfication of the general complex systems is presented in the paper.

In the practical application, the paper provides a general method as well as a basic process for discrimination of the structural types of a system according to the structural digraph. This makes it possible to analyze the complex system model of AHP with computer.

THE (0-1)-MATRIX OF THE STRUCTURE DIGRAPH

When the structure of a system S is expressed in form of digraph G, the research on the reaction between elements of the system is equivalent to that on the connectedness of the digraph. The digraph G is now called the structure digraph of the system S.

It is well known, from the graph theory, that a pair of points in digraph G would be strongly connected, if that pair of points are mutualy reachable; and G would be strong, if and only if each pair of it's points is strongly connected.

The connectedness matrix of digrph G, $A=(a_{ij})$, is a (0-1)-matrix, in which

$$a_{ij} = \begin{cases} 1, & \text{there is a directed arc from point i to point j in G;} \\ 0, & \text{otherwise.} \end{cases}$$

The matrix A is also called the structure matrix of the system S. Obviously, there is a 1-1 correspondence between digraphs and (0-1)-matrices.

According to Boolean calculation, the power $A^{l}=(a_{ij}^{(l)})$ is also a (0-1)-matrix, and

$$a_{ij}^{(l)} = \sum_{k=1}^{n} a_{ik}^{(l-1)} a_{kj}$$

where, n is the number of points in digraph G.

Some concepts relating to the power of the (0-1)-matrix are introduced as follows:

Definition 1: If there is such a power A^m in the sequence $[A^k: k=1,2,\ldots]$ that $A^m=A^{m+1}$, the matrix A is called convergent; if, in the sequence above, there is such a power A^k that $A^k=A^{k+d}$, the matrix A is called periodic. The number K or d, being the least positive integer which makes $A^k=A^{k+d}$ true, is called the index or the period of matrix A.

As a special case, a (0-1)-matrix is called universal matrix and denoted by J, if all the elements of the matrix are one.

Definition 2: the (0-1)-matrix $R=(r_{ij})$ is called the reachability matrix of the digraph G, where

$$r_{ij} = \begin{cases} 1, & \text{the point } j \text{ is reachable from the point } i \text{ in } G; \\ 0, & \text{otherwise.} \end{cases}$$

From definition, we can get

$$R = I + A + A^2 + A^3 + \cdots + A^{n-1}$$

According to Boolean calculion, we induce that

$$R = (I + A)^{n-1}$$

So far, we have introduced the (0-1)-matrix, which is considered as a powerful tool in structure analysis. In the later discussion, the meaning of the letters is specified as follows:

S: the complex system;
A: the structure matrix of S;
G: the structure digraph of S;
R: the reachability matrix of G;
D: the point set of G;
n: the number of points of G;
k: the index of A;
d: the period of A.

THE STRUCTURAL FEATURE OF REDUCIBLE SYSTEM

In the theory of non-negative matrix, the concept of reducibility is of important value to the analysis and classification of the matrix. We can also introduce the same concept in our case.

Definition 3: The structure matrix A, of digraph G, is called irreducible if there is not such a nonempty proper subset in the point set D of G that $a_{ij}=0$ for any $i\in D$ and $j\in\bar{D}$.

The following lemmas are useful to discuss the irreducible structure further:

Lemma 1: It would be true that $a_{kj}=0$ for $k\in D$ which makes $a_{ik}^{(m)}=1$ (where, m is some integer which is not larger than n), if there are such $i,j \in D$ that makes $a_{ij}^{(l)}=0$ for any $l=0,1,2,\ldots,n-1$.

Proof: Assume it is not true. Then, there is some $k^* \in D$, $k^* \neq i,j$ which suffice $a_{ik^*}^{(l)}=1$ and $a_{k^*j}=1$, but

$$a_{ij}^{(l)} = \sum_{k}^{n-1} a_{ik}^{(l-1)} a_{kj}$$

$$= \sum_{k \neq k^*}^{n-1} a_{ik}^{(l)} a_{kj} + a_{ik^*}^{(l)} a_{k^*j}$$

$$= \Sigma + 1 = 1$$

This is contrary to the condition $a_{ij}^{(l)} = 0$.

Lemma 2: Matrix A is irreducible if and only if there is such a $m_{ij} \leqslant n-1$, correspondent to each element a_{ij} of A, that $a_{ij}^{(m_{ij})} = 1$ in power $A^{m_{ij}}$.

Proof: Suppose that the conclusion is not true. i.e., there is some a_{ij} which, for any $l=0,1,2,\ldots,n-1$, makes $a_{ij}^{(l)}=0$. Consider the sets

$$V = [\ x:\ x \in D;\ \text{and for some } m,\ a_{ix}^{(m)}=1]$$

$$\bar{V} = [\ y:\ y \in D;\ \text{and for any } l,\ a_{iy}^{(l)}=0\ ]$$

where, $1 \leqslant m$, $1 \leqslant n-2$.

According to lemma 1, we can induce that $a_{ij}=0$ for any $i \in V$, $j \in \bar{V}$. And this is contrary to definition 3.

Based on the above lemmas, we can reach the following conclusions:

Theorem 1: The following statements are equivalent.
(1) The structure matrix A is irreducible;
(2) The digraph G is strong connected;
(3) The reachability matrix R=J.

We prove the equivalence of these conditions in a cyclic manner.

(1) Implies (2): Suppose G is not strong. Then, there must be such a pair of point $i,j \in D$ that there is no directed path in G witch is from i to j, i.e., $a_{ij}^{(m)}=0$ for any $m=0,1,\ldots,n-1$. But this, according to lemma 2, is contrary to statement(1).

(2) Implies (3): Since G is strong, for each pair of $i,j \in D$ there must be some $m \leqslant n-1$ such that $a_{ij}^{(m)}=1$. Hence $r_{ij}=\sum_{m=0}^{n-1} a_{ij}^{(m)}=1$ i.e., $R=(r_{ij})=J$.

(3) Implies (1): For any $i,j \in D$, since $r_{ij}=\sum_{m=0}^{n-1} a_{ij}^{(m)}=1$ is true, there must be some m^* $(0 \leqslant m^* \leqslant n-1)$ such that $a_{ij}^{(m^*)}=1$, and this shows, according to lemma 2, A is irreducible.

Theorem 1 indicats the relationship between the irreducibility of matrix and the connectedness of digraph. This makes it possible to discuss the classification of strong connected structure further based on the study of the characteristics of irreducible (0-1)-matrix. At the same time, theorem 1 provides a efficient method for discriminating a strong connected structure from it's reachability matrix in practice area.

Now, we prove two theorems which deal with the convergence of the structure matrix of a irreducible system. Before that, we'll introduce some lemmas, first. The proving of these lemmas can be found in the references [2] and [3].

Lemma 3: If the power A^e of a irreducible matrix A is idempotent, then, we have $A^e \geqslant I$ and $e \geqslant k$.

Lemma 4: Suppose $t \geqslant k$. It is true that $a_{jj}^{(t)}=1$ for any $j \in D$, if $a_{ii}^{(t)}=1$ for some $i \in D$.

Lemma 5: Suppose $t \geqslant k$. A^t is idempotent if $A^t \geqslant I$.

Lemma 6: If (0-1)-matrix A is irreducible, the greatest common divisor of $[x: x>0, a_{ii}^{(x)}=1$ for some $i \in D]$ equals that of $[x: x \geqslant k, a_{ii}^{(x)}=1$ for some $i \in D]$.

Lemma 7: Suppose K is the index of a irreducible matrix A, and d is the period of A, then, formula $\sum_{i=1}^{d} A^{r+i}=J$ is always true for any $r \geqslant K$.

Proof: Since A is irreducible and d is the period of A, it can be induced that $R=\sum_{i=0}^{n-1} A^i=J$ and

$$\sum_{i=0}^{n-1} A^i = I + A + A^2 + \cdots + A^{n-1}$$

$$= A^{k+1} + A^{k+2} + \cdots + A^{k+d}$$

$$= \sum_{i=1}^{d} A^{k+i}$$

i.e., $\sum_{i=1}^{d} A^{k+i} = J$.

Theorem 2: Suppose A is a irreducible matrix. The period of the matrix A is the greatest common divisor (GCD) of all x which make $a_{ii}^{(x)}=1$ for any $i \in D$.

Proof: Suppore $x>K$, we have $A^x=A^{x+d}$ (1)
since $a_{ii}^{(x)}=1$, according to lemma 4, it can be induced that $a_{jj}^{(x)}=1$ for any $j \in D$, i.e., $A^x \geqslant I$. This induce that, according to lemma 5, A^x is idempotent. i.e., $A^x=A^{x+x}$ (2)
Comparing formula (1) with (2) and considering the definition of d, we conclude that $d|x$.

As a special case, a irreducible matrix A is called primitive, if the period of A is one.

Theorem 3: An irreducible matrix A is primitive if and only if A is convergent to J.

Proof: Since the period of A is one, it is true that $A^t=A^{t+1}$ for any $t \geq K$. Considering lemma 7, we induce $A^t=A^{t+1}=\cdots=J$, i.e., A is convergent to J. On the other hand, since A is convergent to J, there must be such a m that $A^m=A^{m+1}=\cdots=J$ and this shows the period of A is one.

By the approach of graph theory, the theorems above show that a digraph G, of which the structure matrix A is irreducible, is strongly connected. In digraph G, if some basic circles of different lengths are involved, the GCD of the lengths of all basic circles in G is equal to the period of the system, i.e., the period of matrix A.

As a special case, if the GCD of the lengths of all the basic circles is one, the system, or the structure matrix A, is primitive.

THE ANALYSIS AND CLASSIFICATION OF A REDUCIBLE SYSTEM

It is clear that the structure matrix A of a system is reducible if it's reachability matrix R is not equal to J.

Then the digraph of the system is not strongly connected. To discuss the characteristics of a reducible system further, we introduce following definition first.

Definition 4: A subgraph of a digraph G is called a strong component of G if the subgraph is strongly connected and is not properly include in any other strongly connected subgraph of G.

Definition 5: We can get a new digraph G* based on digraph G by considering each strong component of G as a single point of G* and there is a directed arc in G* which from point v_i to point v_j if and only if there is a directed arc in G which from some point of ith strong component to some point of jth strong component of G. Then, G* is called condesation of G.

It is not difficult to induce that the condesation of a digraph G is unique, and there is no directed path in G*.

If the structure matrix A of a system G is reducible, it can be reformed under the following program:
1. Find out the condesation of G, i.e., the digraph G*.
2. Define the order of the elements of G* with the program introduced in reference [6], p27.
3. Reform the rows and columns of matrix A according to the new order in 2.

So, we get the reformed structure matrix of G witch has the form of:

$$\bar{A} = \begin{bmatrix} c_1 & & N \\ & c_2 & \\ & & \ddots \\ 0 & & c_s \end{bmatrix} \qquad (3.1)$$

where, c_i, $(i=1,2,\ldots,s)$ are all irreducible square submatrices corresponding to each strong component of G, or 0.

Such a matrix as $\tilde{A}$ is called standard form of the reducible matrix A.

Now, we discuss the relationship between the convergence of matrix A and the structure of the strong component.

Lemma 8: If e_i^T is a column vector in which the ith element is one and all others are 0, k_i is the least positive integer which makes vector $e_i^T A^{k_i}$ appear in the power order for unlimited times, and d_i is the least positive integer which makes $e_i^T A^{k_i+d_i}=e_i^T A^{k_i}$ true, then, d is the least common multiple (LCM) of all d_i.

Proof: Since $A^{K+d}=A^K$, the formula

$$e_i^T A^{K+d}=e_i^T A^K \tag{1}$$

is always right for any i, and $K \geq k_i$. Assume there is some i such that $d_i \nmid d$, i.e.,

$$d=fd_i+r_i \tag{2}$$

where, f, r_i are positive integer; and $0<r_i<d_i$. Then, substituting (2) into (1) we get

$$e_i^T A^{K+fd_i+r_i} = e_i^T A^K \tag{3}$$

but

$$e_i^T A^{K+fd_i+r_i} = e_i^T A^{(K+r_i)+fd_i} = e_i^T A^{K+r_i} \tag{4}$$

comparing (3), (4), we induce that r_i is the least value of d in (1), and this is contrary to the definition of d_i. So $d_i | d$ for any i.

On the other hand, if there is some $\bar{d}$ so that $d_i | \bar{d}$ for any i, it can be induced that $A^{K+\bar{d}}=A^K$. According to the definition of d, we have $\bar{d} \geq d$.

Lemma 9: In the standard form (3.1), for any pair of i.j., the least positive integer d_{ij}, which makes

$$e_i^T A^{K_i+d_{ij}} e_j=e_i^T A^{K_i} e_j$$

true, must integrally divide the period of some irreducible submatix c_K $(K=1,2,\ldots,s)$

Proof: Suppose the conclusion of lemma 9 is not true. According to $e_i^T A^{K_i+d_{ij}} e_j=e_i^T A^{K_i} e_j$, there must be a directed circle of length d_{ij} which is from point i to point i. But d_{ij} can not integrally divide the period of any c_K $(K=1,2,\ldots,s)$, the submatrix of the standard form (3.1). Therefor, this circle is a new strong compotant. But this is impossible.

Theorem 3. The period of the structure matrix of a reducible system is equal to the LCM of the periods of all the strong components of the system. This theorem can be induced immediately from the lemma 8 and 9

Corollary: The LCM of all strong components of the reducible system equals one, if and only if the irreducible submatrices c_K in standard form (3.1) are all primitive. Then, the structure matrix is proper and convergent to a limited matrix which is not equal to 0.

If the reducible system has no strong components, the digraph of the system has no directed circle. Hence it's structure matrix is convergent to 0. Then there is not feed back in the system.

PROGRAM FOR ANALYSING THE STRUCTURE OF A GENERAL SYSTEM

As a result, we summarize the conclusions above and get the general procedure of structural analysis in figure 1.

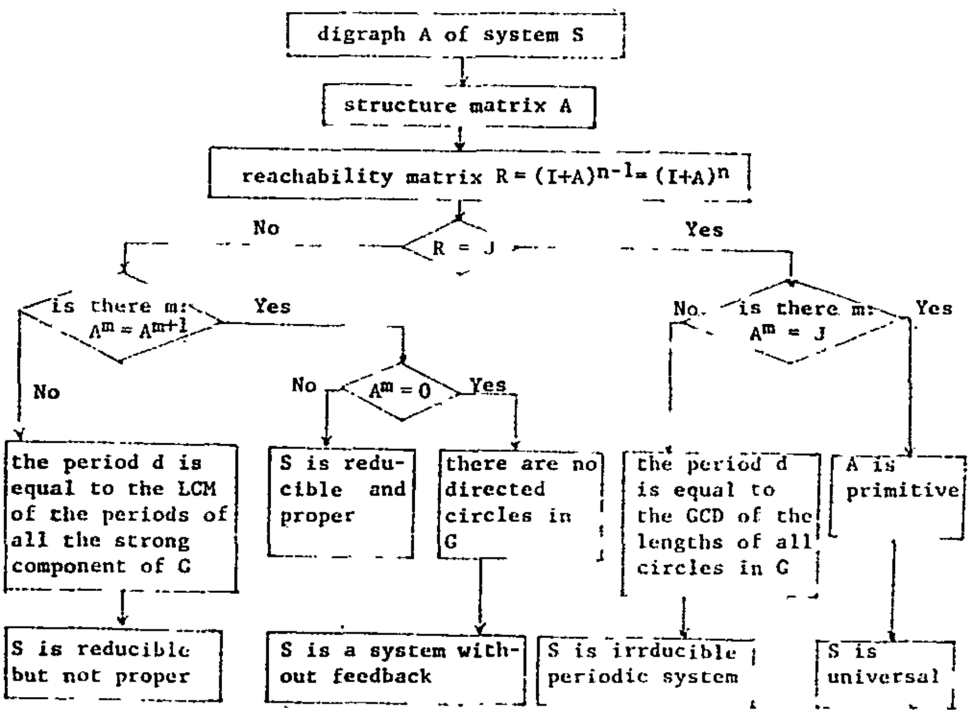

Figure 1

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