

PRODUCT AHP AND ITS PROPERTIES

LIU Qizhi

Information school
Renmin University of China
Beijing P.R.China
liuqz@ns.cetin.net.cn

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Summary: *In this paper we have described Product AHP steps, analyzed its hierarchy structure character, product synthesis character, rank preservation property and alternative set open property, stressed that under a quite general condition the Product Method is the only rank preservation method.*

1. Introduction

The Analytic Hierarchy Process (AHP) proposed by T.L. Saaty in 1970's has been widely used in corporate planning, portfolio selection, and benefit/cost analysis. It enables us make decision systemically, rationally and quantificationally.

But there are some areas in which the AHP has been criticized from the beginning. One of them is rank reversal [15][16][22]. Many papers discussed how to preserve rank. Product Method is a revision method of AHP with which rank reversal can be solved thoroughly.

The Product Method was proposed in 1988[2] [1]. It has many good properties. In 1995, Liu[3] had proved that under some condition the Product Method is the only rank preservation method furthermore. Although a number of papers [4][5][6][7][8] mentioned the Multiplicative AHP which is similar to Product AHP, but seldom paper discussed this method deeply. In this paper we describe the procedure, foundation theory of Product Method and analyze its properties.

2. The Product AHP Procedure

The Product Method can be generalized as four steps: structuring hierarchy, constructing reciprocal matrices, setting priorities of subcriteria, and synthesizing hierarchy results.

2.1 Structuring hierarchies

The procedure of structuring hierarchies is the same to traditional AHP, but the concepts are different. The goals, factors named *criteria*. Between any two criteria A and B, if B has effect on A directly, then it is defined A *dominates* B; B is the *subcriterion* of A.

Taking the criteria as *points* and domination relation as *directed edges*, there is a *directed graph*. The AHP use a special directed graph---one root and no circle directed graph. For any node *n*, the longest length of the path from the root to *n* defined the number of the layer which node *n* is in. For a node *n*, if it dominates nodes *n1* and *n2*, *n1* and *n2* are not in the same level, we can add some virtual nodes and edges revising the graph to satisfy that *n1* and *n2* are in the same level. So not lost generality we can assume that any node *n* in the graph is just dominated by the same level nodes and the nodes which *n* dominating are in one level. So the hierarchy structure is like in figure 1:

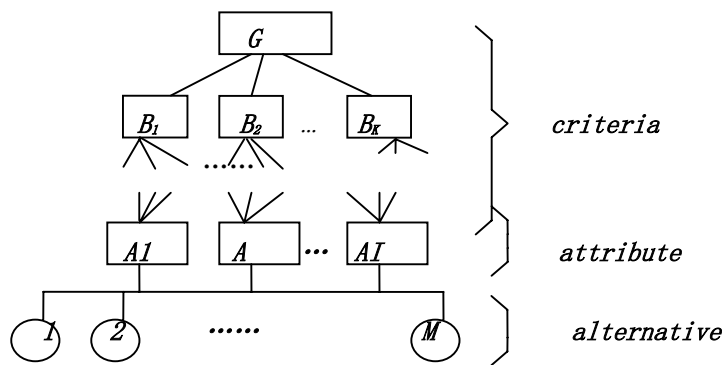


Figure 1

In Product Method the lowest level is distinguished from the criteria, it is named *alternative level*. The level just above the alternative level is named *attribute level*.

2.2 Constructing reciprocal matrices by paired comparisons with ratio scales

After developing the hierarchy, for every criterion, decision-maker need establish the relationships between its subcriteria by comparing in pairs. When the subcriteria being compared, the 1-9 Scale Table can be used.

1-9 Scale Table	
Scale	Definition
1	Equal importance
3	Moderate importance
5	Essential or strong importance
7	Very strong importance
9	Extreme importance
2,4,6,8	Intermediate values between the two adjacent judgment
Reciprocals	If subcriterion U_i has one of the above numbers assigned to it when compared with subcriterion U_j , then U_j has the reciprocal value when compared with subcriterion U_i

Suppose U_1, U_2, \dots, U_M , are subcriteria of criterion C , for any two subcriteria U_i and U_j , the decision-maker should judge by ratio scales which one is more important. The result of pairwise comparisons is a positive reciprocal matrix.

Such pairwise comparison is repeated for all the criteria in each level.

The procedure is the same to traditional AHP.

2.3 Setting priorities of subcriteria dominated by one criterion

The pairwise comparison matrix is obtained by local observation. The essential priority relationship of the subcriteria can be developed from the matrix. The priority relationship is a positive vector; it is named *priority vector* (or *weighted vector* or *relative importance vector*). Usually the vector is normalized, that is mean the sum of the vector elements is 1.

There are many methods to calculate the priority vector from the reciprocal matrix, such as Eigenvalue Method, Least Squares Method, and Least Logarithmic Squares Method etc.

The consistency of the pairwise comparison matrix must be tested, if the consistency ratio (C.R.) of C.I. to that from the matrix is significantly small, the matrix passes the consistency test, and otherwise the matrix should be improved.

This step is the same to traditional AHP.

2.4 Synthesizing hierarchy results

2.4.1 The measurement of criterion importance and the ruler of synthesis

The criterion importance value is defined recursively as following:

Suppose A_1, A_2, \dots, A_m are subcriteria of a criterion B_i ; $(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{im})$ is the priority vector of subcriteria A_1, A_2, \dots, A_m ; a_1, a_2, \dots, a_m are the importance values of subcriteria A_1, A_2, \dots, A_m for an alternative respectively. let bi be the **importance value** of criteria B_i for this alternative, then bi is defined as:

$$bi = a_1^{\alpha_{i1}} \times a_2^{\alpha_{i2}} \times \dots \times a_m^{\alpha_{im}} \quad \dots\dots(2.1)$$

For the criteria in the attribute level, the importance values (**attribute values**) can be obtained from the reciprocal matrix through step 2.2 and step 2.3, or the attribute values can be obtained by DELPHY method, measurement from alternative directly. For one attribute, it is not necessary to make the elements of all alternatives to be normal (the sum of the elements equal to 1). The different ways can be served simultaneously for different attributes respectively in one decision-making. For convenience and not lost generality, the components of attribute values are restricted to positive numbers, the greater the value is, the better the quality will be. If an attribute does not agree with this property, the value can be transformed to obey the stipulation.

2.4.2 Bottom up synthesis and top down synthesis

There are two ways to synthesize, **bottom up** and **top down**.

For any alternative, with bottom up synthesis the criteria importance values can be obtained by formula (2.1) recursively from bottom level up to the general criterion.

The general criterion importance value can be written as:

$$x_1^{\gamma_1} x_2^{\gamma_2} \dots x_n^{\gamma_n} \quad \dots\dots(2.2)$$

The value has two parts, the first is the set of the attribute values, it is a vector (x_1, x_2, \dots, x_n) . The second is the set of the powers respecting to the attribute importance values. It is also a vector $(\gamma_1, \gamma_2, \dots, \gamma_n)$.

Top down synthesis calculates the power vector faster. The priority vectors in the same level can be ranged as a **priority matrix**. The power vector can be obtained by multiplying priority matrices.

The prefer order can be obtained by ranking general criterion importance values of the alternatives.

3. Example

If we reduce figure 1 to four levels, general criterion G , criteria B_1, B_2 subcriteria, A_1, A_2, A_3 and alternative level (figure 2).

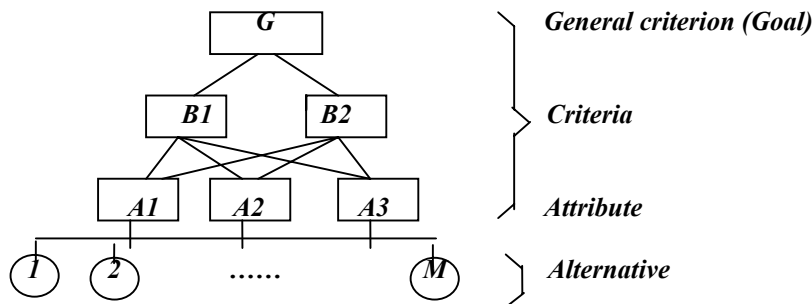


Figure 2

Suppose $(\alpha_{11}, \alpha_{12}, \alpha_{13})$ is the priority vector of subcriteria A_1, A_2, A_3 dominated by criterion B_1 ; $(\alpha_{21}, \alpha_{22}, \alpha_{23})$ is the priority vector of subcriteria A_1, A_2, A_3 dominated by criterion B_2 ; (β_1, β_2) is the priority vector of criteria B_1, B_2 dominated by general criterion G ; (a_1, a_2, a_3) is the importance value vector of subcriteria A_1, A_2, A_3 for an alternative a .

If we use Product Method to deal with this example, for the alternative a , the importance value of general criterion G is

$$a_1^{c_1} \times a_2^{c_2} \times a_3^{c_3}$$

If we use traditional AHP to deal with this example, for the alternative a , the importance value of general criterion G is

$$c_1 a_1 + c_2 a_2 + c_3 a_3$$

Priority vector (c_1, c_2, c_3) is the same, it can be obtained by multiplying priority matrices.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \\ \alpha_{13} & \alpha_{23} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

The prefer order of alternatives is obtained by ranking general criterion importance values.

4 The characters and properties of the Product Method

4.1 The formula of general criterion importance value

If we use a vector (x_1, x_2, \dots, x_n) to define the attribute values of an alternative, the general criterion importance value of this alternative can be written as a function $f(x_1, x_2, \dots, x_n)$. The function f represents the way and preference of decision-maker. In AHP the function $f(x_1, x_2, \dots, x_n)$ is related with the priority (or weighted) vector $(\gamma_1, \gamma_2, \dots, \gamma_n)$. This vector is decided by the hierarchy structure, it is named **hierarchy structure vector**.

In traditional AHP,

$$f(x_1, x_2, \dots, x_n) = \gamma_1 x_1 + \gamma_2 x_2 + \dots + \gamma_n x_n$$

In Product AHP,

$$f(x_1, x_2, \dots, x_n) = x_1^{\gamma_1} \times x_2^{\gamma_2} \times \dots \times x_n^{\gamma_n}$$

In AHP, any hierarchy can be treated as three levels: general criterion, attribute level and alternative level.

4.2 Hierarch structure and synthesis characters of Product Method

When we use Product Method to make decision, there are two characters contrasting with traditional AHP. One is in the hierarchy structure the other is in synthesis.

Although the construction of the hierarchy in Product Method is similar with that in traditional AHP, but the cognition of the hierarchy is difference. In traditional AHP the alternative level is treated as common criterion level and the synthesis ruler (from subcriteria to criterion) must be uniform. In Product Method alternative level is separate from criteria, the criterion synthesis ruler and the ruler of calculating attribute value may be different. When we synthesize from subcriteria to criterion the traditional AHP uses sum of weighted subcriteria, contrast, Product AHP uses product of powered subcriteria.

The product synthesis way has been mentioned in [5][6][7], it is named multiplicative AHP. Because the synthesis ruler for all levels (including alternative level) must be uniform in traditional AHP, therefore the multiplicative AHP cannot extend to Analytic Network Process (ANP). That is a main reason of Saaty refusing multiplicative AHP. Product AHP has overcome this fault. It is easy to extend Product AHP to ANP. We shall discuss the Product ANP in [28].

4.3 The property of rank preservation

In AHP, for any two alternatives, suppose the attribute vectors are $((x_1(1), x_2(1), \dots, x_n(1)))$ and $((x_1(2), x_2(2), \dots, x_n(2)))$ respectively.

Property 1. The ratio of two alternative importance values is not changed when any attribute values multiply same constant in Product Method. That is

$$\frac{f(k_1x_1(1), \dots, k_nx_n(1))}{f(k_1x_1(2), \dots, k_nx_n(2))} = \frac{f(x_1(1), \dots, x_n(1))}{f(x_1(2), \dots, x_n(2))}$$

In Product Method if the attribute value is obtained by paired comparisons of alternatives, then the ratio of two alternative importance values does not change, when adding or deleting alternative. If the attribute value is obtained by measurement directly, then the alternative importance values do not change when adding or deleting alternative. So adding or deleting alternative does not affect the alternative priority order. We have proved:

Corollary 1. In Product Method, adding or deleting alternative does not change the alternative priority order

From Corollary 1 we can see that rank reversal in relative measurement [16] will not occur in Product AHP.

From Property 1, the ratio of two alternative importance values does not change when the measurement unit of an attribute is changed. So we have:

Corollary 2. In Product Method, attribute values enlarging or reducing linearly does not change the alternative priority order.

From Corollary 2 we also can see that rank reversal in absolute measurement [1][2] will not occur in Product AHP.

There is a theorem that plays a very important role when we discuss rank reversal and rank preserve in Product AHP.

Theorem. If f is a continue function, not always be 0, then for all

$k_i > 0, x_i(1) > 0, x_i(2) > 0, i=1, 2, \dots, n$

$$\frac{f(k_1x_1(1), \dots, k_nx_n(1))}{f(k_1x_1(2), \dots, k_nx_n(2))} = \frac{f(x_1(1), \dots, x_n(1))}{f(x_1(2), \dots, x_n(2))} \quad \dots\dots (4.1)$$

If and only if

$$f(x_1, \dots, x_n) = Ax_1^{\alpha_1} \dots x_n^{\alpha_n} \quad \dots\dots(4.2)$$

$A, \alpha_1, \alpha_2, \dots, \alpha_n$ are all constants.

We leave the proof in next section. Now we clarify what properties the Product Method will have if the theorem is true.

The ratio of two alternative importance values does not change can introduce that the alternatives priority order not changing, but on the contrary it is not true. So the condition of two alternative importance values not changing is stronger than the condition of alternatives priority order not changing. But two alternative importance values not changing is quite a general and convenient condition. From the theorem we see the Product Method is the only method in which the ratio of importance values not changed. So we have:

Property 2. If we strengthen the condition of alternative priority order not changing to be the condition of the ratio of alternative importance values not changing, then the Product Method is the only rank preservation method.

4.4 The open property of alternative set

The decision making process with AHP is the ranking or selecting alternative process by decision-maker's subjective analyses. The attribute vector (x_1, x_2, \dots, x_n) is the quantity of attribute, hierarchy structure vector $(\gamma_1, \gamma_2, \dots, \gamma_n)$ reflect the preference of decision-maker. In traditional AHP we have accepted that assumption: the formula of general criterion importance value just suit to a certain alternative set. For example, in relative measurement when adding or deleting an alternative the decision

problem is changed from one to another. Although the situation in absolute measurement is not obvious, but the matter is the same. Because the alternative attribute values are limited to the fixed numbers. Therefore there is a maximum alternative set in absolute measurement. The rank reversal occurs between different alternative sets. Infected the fault is that: on one side accepting the method just suit for the certain decision-making problem, but on the other side contrasting the results between two problems. In Product Method, the decision function is not changed but the object --alternative set can be opening.

5. The proof of the theorem

Proof:

It is obvious that (4.2) => (4.1) is true. In the following we prove (4.1) => (4.2) is true.

The proof is carried on Mathematical Induction for dimension number n .

First case, when $n=1$ the conclusion is correct.

From the (4.1) is true we have: for all $k_1 > 0, x_1(1) > 0, x_1(2) > 0,$

$$\frac{f(k_1 x_1(1))}{f(k_1 x_1(2))} = \frac{f(x_1(1))}{f(x_1(2))} \quad \dots\dots(5.1)$$

Suppose $f(1) \neq 0$, in (5.1). Let $x_1(2) = 1, k_1 = u, x_1(1) = v$, then for all $u > 0, v > 0$ we have:

$$f(uv) = f(u)f(v)/f(1)$$

then if $x > 0$ for any natural number p we have:

$$f(x^p) = f(1)(f(x)/f(1))^p$$

Let $x^p = z$ then

$$f(z) = f(1)(f(z^{1/p})/f(1))^{1/p}$$

That is $f(z^{1/p}) = f(1)(f(z)/f(1))^{1/p}$

So for any positive rational number p/q , we have:

$$\begin{aligned} f(x^{p/q}) &= f(1)(f(x^{1/q})/f(1))^p \\ &= f(1)(f(x)/f(1))^{p/q} \end{aligned}$$

Because f is a continue function, so for any two positive real numbers a, x , we have:

$$f(x^a) = f(1)(f(x)/f(1))^a$$

Especially let $x=2$, denote $y=2^a$

Then $a = \log y / \log 2$ and then

$$\begin{aligned} f(y) &= f(1)(f(2)/f(1))^{\log y / \log 2} \\ &= f(1)(f(2)/f(1))^{\log y}^{1/\log 2} \\ &= f(1)y^{(\log(f(2)/f(1)))/\log 2} \\ &= A y^d \end{aligned}$$

In the above formula $A = f(1), d = (\log(f(2)/f(1)))/\log 2$

So we have proved when $n=1$, (4.1) => (4.2) is true.

Next we prove if (4.1) => (4.2) is true for n , then (4.1) => (4.2) is true for $n+1$.

Let the n -dimension vectors $k, x1, x2$ are defined as:

$$k = (k_1, \dots, k_n)$$

$$x1 = (x_1(1), \dots, x_n(1))$$

$$x2 = (x_1(2), \dots, x_n(2))$$

Denote $k \bullet x = (k_1 x_1, \dots, k_n x_n)$

$y1 = x_{n+1}(1), y2 = x_{n+1}(2), k_{n+1} = a$

Then from (4.1) is true for $n+1$ we have:

$$\frac{f(k \bullet x1, ay_1)}{f(k \bullet x2, ay_2)} = \frac{f(x1, y_1)}{f(x2, y_2)} \quad \dots\dots(5.2)$$

In (5.2), especially let n -dimension vectors $k = (1, 1, \dots, 1), x1 = x2 = x = (x_1, x_2, \dots, x_n)$, then (5.2) can be written as:

$$\frac{f(x, ay_1)}{f(x, ay_2)} = \frac{f(x, y_1)}{f(x, y_2)} \quad \dots\dots(5.3)$$

In (5.3), the function $f(x,y)$ can be deal with as such a function of singular variable of y . Because when $n=1$ the conclusion is correct, so we can get:

$$f(x,y) = f(x,1) \cdot y^{(\log(f(x,2)/f(x,1)))/\log 2} \quad \dots\dots(5.4)$$

Not lost generality, let $f(1, \dots, 1, 1) \neq 0$, because we suppose (4.1) is true for $n+1$, so we have:

$$\frac{f(x,2)}{f(x,1)} = \frac{f(1, \dots, 1, 2)}{f(1, \dots, 1, 1)}$$

And we have supposed (4.1) \Rightarrow (4.2) is true for n , so we have:

$$f(x,1) = Ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

$$\text{Let } \beta = (\log \frac{f(1, \dots, 1, 2)}{f(1, \dots, 1, 1)}) / \log 2$$

Then (5.4) can be written as:

$$f(x, y) = Ax_1^{\alpha_1} \dots x_n^{\alpha_n} \times y^\beta$$

End.

6. Conclusion

In this paper we have described Product AHP procedure, listed its hierarchy structure character, product synthesis character, rank preservation property and alternative set open property, stressed that in a quite general condition the Product Method is the only rank preservation method.

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