SOME NOTES ON THE CONSISTENCY CHECKS OF THE AHP WITH DISCRETE SCALES

Su Tingao Hu Engkhe Wang Chunmei
Inner Mongolia University
Huhhot, China

ABSTRACT

The consistency checks of judgement matrices are both an important part of the theory of the AHP and the necessary steps in the applications of it. In the first half of the paper the consistency checking methods put forward respectively by T.L. Saaty and Xu Shubo are discussed and their approximateness and limitation are analyzed. In the latter half, a concept of semi-random judgement matrix of grade k and its special case ---- quasi-consistent matrix are introduced. After distinguishing the two kinds of reasons i.e. the judgement faults and the limitation of discrete scales, which often cause inconsistencies, starting from improvement of subjective judgement, we go for into the approach to refine Saaty and Xu's critical values of consistency checks.

1.INTRODUCTION

The concept of consistency of judgment matrices was proposed by T.L.Saaty when his Analytic Hierarchy Process was coming out .The consistency checking problem is of great importance both to the theory and to its applications.Starting from the Eigenvalue Method in searching the derived scales,Saaty(1980) put forward a method which is calld CR checking method. Let RI(n) be the sample mean of a random variable $\mu=(\lambda_{max}-n)/(n-1)$ resulting from a number of judgment matrices. Saaty takes the value $\bar{\mu}(n)=0.10\cdot RI(n)$ as the critical value and considers those judgment matrices to possess acceptable consistency, in which the consistency indices are equal to or less than $\bar{\mu}(n)$. The critical values improved by Xu Shubo et ål are listed in Table 1.

Table 1. Critical values $\bar{\mu}(n)$ of CR checking

n	3	4	5	6	7	8	9
$\bar{\mu}(n)$	0.0515	0.0893	0.1119	0.1249	0.1345	0.1420	0.1462
n	10	11	12	13	14	15	
$\bar{\mu}(n)$	0.1487	0.1516	0.1541	0.1558	0.1578	0.1589	

Starting from the Logarithmic Least Squares Method (LLSM), Crawford (1987) put forward the s^2--checking which is equivalent to CR checks. Because of the reasoning process in which the critical values $\bar{\mu}(n)$ are used , the s^2--checking has the same merits and demerits as the old checking.However, the literature has advocated the use of CR checking for its simplicity and easiness to operate. But some people considered that the checking critical values are too tolerant for the matrices of lower order and too strict for the matrices of higher order. Xu Shubo (1987b)deems that Saaty's critical values lack rigorous theoretical ground and he presents a statistical method--Chi-square checking used in an individual prioritization. In our analysis we always take the scale set which consists of integers 1 to 9 and their reciprocals as routine discrete scale set.After reformulating Xu's results, we shall analyze their approximateness and limitation in Sec.2.Then in Sec.3 we shall introduce the notion of semi-random matrix with which a new improved checking method--Contrast checking is set up. A brief discussion about the significance and shortcomings of the results we obtained constitutes the final section.

The concepts, notations and formulae used in the paper are as follows:

Suppose that $A=(a_{ij})$ is a positive reciprocal matrix of order n, λ_{max} is its principal right eigenvalue. From the theory of non-negative matrices we have $\lambda_{max} \geqslant n$ and λ_{max} is simple. A is a consistent matrix if and only if $\lambda_{max}=n$.Naturally, we take

$$\mu = \frac{\lambda_{max}-n}{n-1} \quad (1)$$

as the consistency index. Letting $w=(w_1,w_2,\ldots,w_n)^T$ be the normalized principal right eigenvector, all the w_is are positive. Let $W=(w_i/w_j)$, $E=(\varepsilon_{ij})=A\circ W^T=(a_{ij}.w_j/w_i)$, then

$$A=E\circ W=(\varepsilon_{ij}.w_i/w_j) \quad (2)$$

where the symbol "o" represents Hadamard product of matrices. Let

$$\Delta=(\delta_{ij})=(\varepsilon_{ij}-1) \quad (3)$$

From Xu Shubo (1987a) we have

$$\mu \doteq -1+\frac{1}{n(n-1)}\sum_{1\leqslant i<j\leqslant n}(1+\delta_{ij}+\frac{1}{1+\delta_{ij}}). \quad (4)$$

2.CHI-SQUARE CHECKING

According to Saaty's conjecture about μ which possesses an approximate Chi-square distribution, Xu Shubo (1987b) put forward a statistical checking method of the consistency of an individual prioritization--Chi-square checking.The results can be formulated as follows:

Theorem.Suppose that the δ_{ij}s are sufficiently small,independent and $\delta_{ij}\sim N(0,\frac{1}{2})$ for all i,j satisfying $1\leqslant i<j\leqslant n$.Then the random variable $2n(n-1)\mu$ has anapproximate Chi-square distribution with freedom of $\frac{n(n-1)}{2}$.

From the theorem we conclude that a sequence of critical values $\tilde{\mu}(n)$ at the confidence level of 90% are listid in Table 2.

Table2. $\tilde{\mu}(n)$- critical values in Chi-square checking

n	3	4	5	6	7	8	9
$\tilde{\mu}(n)$	0.049	0.092	0.122	0.142	0.161	0.169	0.178
n	10	11	12	13	14	15	
$\tilde{\mu}(n)$	0.185	0.191	0.198	0.200	0.204	0.208	

Xu Shubo confirmed that the results above are better than Saaty's empirical values. In fact, its consistency ratios $CR_x(n)=\tilde{\mu}(n)/\bar{\mu}(n)$ listed in Table 3 show that the analysis of Xu's is reasonable.

Table 3. $CR_x(n)$

n	3	4	5	6	7	8	9
$CR_x(n)$	0.0950	0.1030	0.1090	0.1137	0.1197	0.1190	0.1218
n	10	11	12	13	14	15	
$CR_x(n)$	0.1244	0.1260	0.1272	0.1283	0.1293	0.1309	

For the convenience of use, we suggest the following critical values of the consistency ratio

$$CR_x=\begin{cases}0.10 & n=3,4\\ 0.11 & n=5,6\\ 0.12 & n=7,8,9,10\\ 0.13 & n=11,12,13,14,15\end{cases} \tag{5}$$

It is shown from above that taking the value 0.10 as the critical value of consistency ratio is somewhat harsh for the matrices of order $n\geqslant 5$ and to some extent the result obtained tallies partly with the people's conjecture. Thus it can be comfirmed that under some distribution assumption of perturbation, Xu's checking is a better method with some rigorous theoretical basis and probability guarantee.But there are two remarks being worth pointing out.

1) The author adopted the following approximate treatment in his proof:

$$\frac{1}{1+\delta_{ij}}=1-\delta_{ij}+\delta^2_{ij}-\frac{\delta^3_{ij}}{1+\delta_{ij}}\approx 1-\delta_{ij}+\delta^2_{ij} \tag{6}$$

from which the approximate expression is derived

$$\mu=\frac{1}{n(n-1)}\sum_{i,j=1}^{n}\delta_{ij}=\frac{1}{n(n-1)}\sum_{i\neq j}\delta_{ij}\approx\frac{1}{n(n-1)}\sum_{1\leqslant i<j\leqslant n}\delta^2_{ij} \tag{7}$$

2) Because of the confinement of discrete scales, the assumption that the mean of the δ_{ij}s equals zero is also approximate.At The Computation Center of Inner Mongolia university, we obtained the averages of δ_{12} and δ_{13} for the quasi--consistent matrix class* of order 3 using a sample size of 1000, the results of which are as follows:

$$\bar{\delta}_{12}=0.02844,\quad \bar{\delta}_{13}=0.05176 \tag{8}$$

*The precise definition of the class is given in Sec.3.

It is shown simultaneously that the assumption that all δ_{ij}s possess the same distribution is also approximate.

3. THE CONTRAST CHECKING

Both Saaty(1980) and Xu Shubo(1987a) deem that the inconsistency of a judgment matrix is caused by the complexity of objective reality and the diversity of human cognition. In fact, besides the two reasons mentioned above, there is another one causing the inconsistency--the limitation of discrete scales. Let $S=\{1/9,1/8,\ldots,\frac{1}{2},1,2,\ldots,8,9\}$ be the commonly used scale set, whose elements are denoted succesively by notations u_1, $u_2,\ldots,u_{17}$ for convenience.

Definition 1. A square matrix $A=(a_{ij})$ of order n is called a randomly evaluated matrix, if all a_{ij} satisfy

1). $a_{ii}=1$ for all $i=1,2,\ldots,n$;

2). a_{ij} is a scale value drawn randomly from S with replacement for all i,j satisfying $1\leqslant i<j\leqslant n$;

3). $a_{ij}=1/a_{ji}$ for all i,j satisfying $1\leqslant j<i\leqslant n$.

Both Saaty and Xu studied the approximate expectation of the consistency index of this class of matrices.

If a judgment is the best one, there must be a consistent judgment matrix to correspond it. Because of the consistency, there are only n-1 independently evaluated entries in the upper triangular part of the matrix, and the other entries can be derived from the transitivity relation $a_{ij}=a_{ik}a_{kj}$. Without loss of generality, we assume that the independent elements are those of row 1. Now the formal definition of the class mentioned above is given as follows:

Definition 2. A positive reciprocal matrix $A=(a_{ij})$ of order n is called a quasi-consistent matrix, if each of the upper triangular elements a_{ij} satisfies

1). a_{1j} is a scale value drawn randomly from S with replacement for all $j=2,\ldots,n$;

2). Let $a_{ij}=u_{t_0}$ for i,j satisfying $1\leqslant i<j\leqslant n$, where $u_{t_0}\in S$ satisfies

$$\left|u_{t_0}-a_{i1}a_{1j}\right|=\min_{1\leqslant t\leqslant 17}\left\{\left|u_t-a_{i1}a_{1j}\right|\right\}.$$

Evidently,a quasi-consistent matrix is also an inconsistent matrix ,but its inconsistency is only caused by the limitation of the discrete scales. After the set is determined, the consistency obtained is an impassable one. Thus the quasi-consistent class Q of matrices is the best class among all the inconsistent matrices.We computed the sample mean μ_q and the variance σ_q^2 using a sample size of 1000 for matrices of order 3 through 15. The results are as follows:

Table 4. μ_q and σ_q^2 of class Q

n	3	4	5	6	7	8	9
$\mu_q(n)$	0.0381	0.0531	0.0588	0.0650	0.0669	0.0682	0.0692
$\sigma_q^2(n)$	0.0041	0.0030	0.0021	0.0015	0.0011	0.0009	0.0007
n	10	11	12	13	14	15	
$\mu_q(n)$	0.0699	0.0706	0.71	0.0711	0.0715	0.0719	
$\sigma_q^2(n)$	0.0006	0.0005	0.0004	0.0004	0.0003	0.0003	

Let $\hat{\mu}$ be a sample mean of consistency indices of judgment matrices with acceptable consistency, then no matter what priciple and method are used to derive the critical values, it should satisfy the inequality

$$\mu_q(n) \leqslant \hat{\mu}(n) \leqslant \bar{\mu}(n) \qquad (9)$$

For example we should have $0.0381 \leqslant \hat{\mu}(3)$, $1.5894 \geqslant \hat{\mu}(15)$ etc. Either cancelling or harshenning the consistency indicator is not reasonable. Though it is not easy to find out a class of matrices with acceptable consistency in which the complexity of reality and the diversity of cognition are taken into consideration, it is relatively easy to find out a contrast class of matrices of which the consistency index is greater than or equal to that of the acceptable class. As a matter of fact, Saaty takes the randomly evaluated matrices as the contrasting class, 10 percent of the average consistency index of it is taken to be the critical value. In this case, however, human's rational thinking is not taken into account completely. On the other hand, if we take the quasi-consistent class as a standards, the complexity of reality and the diversity of cognition are left out of consideration. Therefore we introduce the notion of the semi-random matrix in the following

Definition 3. A positive reciprocal matrix $A=(a_{ij})$ of order n is called a semi-random matrix of grade k, if its upper triangular elements satisfy:

1). a_{1j} is a scale value drawn randomly from S with replacement for $j=2,\dots,n$;

2). For i,j satisfying $1<i<j\leq n$, let $u_r\in S$ be satisfied

$$|u_r-a_{i1}a_{1j}|=\min_{1\leq t\leq 17}\{|u_t-a_{i1}a_{1j}|\}$$

and a_{ij} is a scale value drawn randomly from $S_{r,k}$ with replacement, where

$$S_{r,k}=\{v_{r-k},\dots,v_{r-1},v_r,v_{r+1},\dots,v_{r+k}\}\cap S$$

here
$$v_t=\begin{cases}u_t & 1\leq t\leq 17\\ 0 & t<1 \text{ or } t>17\end{cases}$$

Obviously, when k=16, i.e. a semi-random matrix of grade 16 is a randomly evaluated matrix and when k=o, i.e. a semi-random matrix of grade 0 is a quasi-consistent matrix. For general k, say, k=2, its elements evaluated as follows: if, for example, $a_{12}=3$, $a_{13}=8$, then $|3-8/3|=\min_{1\leq t\leq 17}\{|u_t-a_{21}a_{13}|\}$.
If one goes by the semi-randomness of grade 2, a_{23} is a scale value drawn randomly from a set $\{1,2,3,4,5\}$, i.e. the value of a_{23} is fluctuated at the weak importance with 2 grades up and down.

Comparing with results of Saaty's and Xu's, one can believe that an acceptable inconsistent class should be close to the semi-random class of grade 2. So we take it as our contrasting class. Using sample size of 2000, Su Tingao et al (1988) computed the empirical distribution of $\mu=\frac{\lambda max-n}{n-1}$ for classes of order 3 through 15.

Taking account of the less complexity and diversity of judgment matrices of order 3, we choose $\mu_q(3)=0.0381$, the average consistency index of quasi-consistent class as our critical value $\hat{\mu}(3)$ of acceptable consistency. From the empirical distribution we have $P\{\mu\leq\hat{\mu}(3)\}=0.5075$, precisely the area magnitute of the curvilinear trapezoid under the empirical density curve and on the left of point 0.0381. Using the area-criteria of about 0.5075, just like being used for n=3, we have a sequence of critical values $\hat{\mu}(n)$ and the corresponding areas $P(\hat{\mu})$, for n=3,4,···,15, which are listed in Table 5.

Table 5. $\hat{\mu}(n)$, n=3,4,···,15

n	3	4	5	6	7	8	9
$\hat{\mu}(n)$	0.038	0.098	0.126	0.145	0.162	0.174	0.184
$P(\hat{\mu}(n))$	0.5075	0.512	0.5075	0.5070	0.5025	0.5095	0.5130
n	10	11	12	13	14	15	
$\hat{\mu}(n)$	0.189	0.196	0.200	0.203	0.206	0.208	
$P(\hat{\mu}(n))$	0.5105	0.5135	0.5025	0.5100	0.5110	0.5045	

4. CONCLUSION

Though our contrast checking method still lacks rigorous theoretical foundations, the critical values obtained from contrasting with the semi-random matrices of grade 2 synthesize all the factors descending consistencies of judgmentet matrices,such as the complexity of reality, the diversity of cognition and the restriction of discrete scales. Furthermore, the checking unifies the tolerance standards of consistency checks with a certain intuitive background. So our method is an improvement of Saaty's checking, and it is not happening by chance that our cri tical values tally fairly with Xu's results.

As regards the consistency checks of an entire hierarchy, it remains to be discussed further.

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