Principles of Rank Preservation Graded Eigenvector Method

Weitao Zhang
Mathematical Teaching and Research Section
Northwestern Institute of Architectural Engineering
Xian, Shaanxi 710061
China

Abstract

The Graded Eigenvector Method (GEM) uses the upper triangular part of the matrix of judgments A to derive the priorities of the alternatives. Here we study the principles of rank preservation of GEM under a single criterion when new elements are added.

1. Introduction

Yu [1] and Wang [2] use the matrices N and M given by:

$$N = \begin{bmatrix} 1/n & a_{12} & \cdots & a_{1n} & a_{1n} \\ & 1/(n-1) & & \cdots & a_{12} \\ & & \ddots & & \vdots \\ & O & & 1/2 & a_{1\,n-1} \\ & & & & 1 \end{bmatrix}, M = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & 2 & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & O & & n \end{bmatrix}$$

to derive the priorities of alternatives. In this paper, we use the upper triangulilar matrix M to study the principles of rank preservation under a single criterion when new elements are added to the set of alternatives being compared..

Definition (Strong Rank Preservation): Given an nxn matrix of judgments $A = (a_{ij})$. The rank of the alternatives is strongly preserved if a new element is added to the set and the new resulting (n+1 x n+1) matrix of paired comparisons B satisfies:

$b_{ij} = a_{ij}$ for $i,j = 1,2,\ldots,n$

$b_{i,n+1} / b_{j,n+1} = a_{i,n+1} / a_{j,n+1}$ for all i and j.

Definition (Weak Rank Preservation): Given an nxn matrix of judgments $A = (a_{ij})$. The rank of the alternatives is weakly preserved if a new element is added to the set and the new resulting (n+1 x n+1) matrix B satisfies:

$b_{ij} = a_{ij}$ for $i,j = 1,2,\ldots,n$

2. Strong Rank Preservation

Theorem 2.1[3] Let the judgments of the upper triangular matrix A of order n be consistent. Let $\lambda_{max}(A) = n$ and $W = (W_1, W_2, \ldots, W_n)^T$ be the principal eigenvalue and eihenvector of A, respectively. Let A^* be the upper triangular matrix of order (n+1) given by:

$$A^* = \begin{bmatrix} A & \begin{matrix} a_{1\,n+1} \\ \vdots \\ a_{n\,n+1} \end{matrix} \\ O & n+1 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & 2 & \cdots & a_{2n} \\ & & \ddots & \vdots \\ O & & & n \end{bmatrix}$$

Let $\lambda_{max}(A^*)$ and $W^* = (W^*_1, W^*_2, \ldots, W^*_{n+1})^T$ be the principal eigenvalue and eigenvector of A^*, respectively. When a new element is added, a necessary and sufficient condition for strong rank preservation is that $a_{i\,n+1} = cw_i$, $(c>0)$, $(i=1,2,\ldots,n)$ for the judgment matrix A.

Proof: Let us prove the sufficient principle. Suppose that $a_{i\,n+1} = cw_i$, $(c>0)$, $(i=1,2,\ldots,n)$. Let

$$\lambda_{max}(A) = \lambda,\ \lambda_{max}(A^*) = \lambda+\Delta\lambda,\ \bar{W}^* = (W^*_1, \ldots, W^*_n)^T$$

From the eigenvalue problem:

$$A^* W^* = (\lambda + \Delta\lambda) W^*_{n+1} \tag{1}$$

we obtain:

$$A\bar{W}^* + (a_{1\,n+1}, a_{2\,n+1}, \cdots, a_{n\,n+1})^T W^*_{n+1} = (\lambda + \Delta\lambda)\bar{W}^* \tag{2}$$

$$(n+1)W^*_{n+1} = (\lambda + \Delta\lambda) W^*_{n+1} \tag{3}$$

and we have

$$W^*_{n+1} = \Delta\lambda/c, \quad c>0. \tag{4}$$

Substituting (4) into (2) we have:

$$A\bar{W}^* + (CW_1, CW_2, \cdots, CW_n)^T(\Delta\lambda / C) = (\lambda + \Delta\lambda)\bar{W}^* \tag{5}$$

From (3) we have $\Delta\lambda = 1$, thus $W^* = (W_1, W_2, \cdots, W_n, 1/C)^T$, $(c>0)$. Q.E.D.

Let us now prove the necessary condition. Suppose that after adding a new element, the judgment matrix A^* is strong rank preserving, and the corresponding principal eigenvalue and eigenvector are given by $\lambda_{max}(A^*)$ and $\bar{W}^* = (W^*_1, \ldots, W^*_n, W^*_{n+1})^T$, respectively. Similarly, assign $\lambda_{max}(A) = \lambda$, $\lambda_{max}(A^*) = \lambda+\Delta\lambda$. From the eigenvalue problem (1) we have:

$$A\bar{W}^* + (a_{1\,n+1}, a_{2\,n+1}, \cdots, a_{n\,n+1})^T W^*_{n+1} = (\lambda + \Delta\lambda)\bar{W}^* \tag{6}$$

$$(n+1)W^*_{n+1} = (\lambda + \Delta\lambda)W^*_{n+1} \tag{7}$$

and hence we have $W^{*n+1} = \Delta\lambda/c$, $c>0$. (8)

Substituting (8) into (6) we obtain:

$$A\bar{W}^* + (a_{1\,n+1}, a_{2\,n+1}, \cdots, a_{n\,n+1})^T(\Delta\lambda / C) = (\lambda + \Delta\lambda)\bar{W}^* \tag{9}$$

or $$A\bar{W}^* = \lambda\bar{W}^* + \Delta\lambda[\bar{W}^* - (a_{1\,n+1}/C, a_{2\,n+1}/C, \cdots, a_{n\,n+1}/C)^T] = 0 \tag{10}$$

Evidently, $\bar{w}^* = (a_{1\,n+1}/c, \ldots, a_{n\,n+1}/c)^T$ satisfies (10). Since A^* is a strong rank preserving, we have $\bar{W}^* = (\bar{W}^*_1, \ldots, \bar{W}^*_n)^T = (W_1, \ldots, W_n)^T$, and the result follows. Q.E.D.

$$a_{i\,n+1} = cW_i, \ (i = 1, 2, \cdots, n) \ (c>0).$$

Corollary 1. Let the judgments of the upper triangular matrix A of order n be consistent. Let $\lambda_{max}(A) = n$ and $W = (W_1, W_2, \ldots, W_n)^T$ be the principal eigenvalue and eihenvector of A, respectively. Suppose that the upper triangular matrix of order n+2 is given by A^*:

$$A^* = \begin{bmatrix} A & & a_{1\,n+1} & a_{1\,n+2} \\ & & \vdots & \vdots \\ & \ddots & a_{n\,n+1} & a_{n\,n+2} \\ O & & n+1 & a_{n+1\,n+2} \\ & & & n+2 \end{bmatrix}, \text{where} A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & 2 & \cdots & a_{2n} \\ & & \ddots & \vdots \\ O & & & n \end{bmatrix}$$

$\lambda_{max}(A^*)$ and $W^* = (W^*_1, W^*_2, \ldots, W^*_{n+1})^T$ are the principal eigenvalue and eigenvector of A^*, respectively. If two new elements are added, the the necessary and sufficient condition for strong rank preservation is given by:

$$\begin{cases} a_{i\,n+1} = C_1 W_i (C_1 > 0), (i = 1,2,\cdots,n) \\ a_{i\,n+2} = C_2 W_i (C_2 > 0), (i = 1,2,\cdots,n) \end{cases}, \text{for the judgement matrix A.}$$

Proof: (Sufficiency) Let

$$\lambda_{max}(A) = \lambda,\ \lambda_{max}(A^*) = \lambda + \Delta\lambda,\ \tilde{W}^* = (W^*_1, \ldots, W^*_n)^T.$$

From the eigenvalue problwm (1) we have:

$$A\tilde{W}^* + (a_{1\,n+1}, a_{2\,n+1}, \cdots, a_{n\,n+1})^T W^*_{n+1} + (a_{1\,n+2}, a_{2\,n+2}, \cdots, a_{n\,n+2})^T W^*_{n+2} = (\lambda + \triangle\lambda)\tilde{W}^* \quad (11)$$

$$(n+1)W^*_{n+1} + a_{n+1\,n+2} W^*_{n+2} = \lambda_{max}(A^*) W^*_{n+1} \quad (12)$$

$$(n+2)W^*_{n+2} = \lambda_{max}(A^*) W^*_{n+2} = (\lambda + \triangle\lambda) W^*_{n+2} \quad (13)$$

and hence we have:

$$W^*_{n+1} = \triangle\lambda / C_1, (C_1 > 0), W^*_{n+2} = \triangle\lambda / C_2, (C_2 > 0) \quad (14)$$

Substituting (14) into (11) we have:

$$A\tilde{W}^* + (C_1W_1, C_1W_2, \cdots, C_1W_n)^T(\triangle\lambda / C_1) + (C_2W_1, C_2W_2, \cdots, C_2W_n)^T(\triangle\lambda / C_2) = (\lambda + \triangle\lambda)\tilde{W}^* \quad (15)$$

Evidently, $\tilde{W}^* = 2W$ is a solution that satisfies (15). Thus, A^* is strong rank preserving. On the other hand, from (13) we have $\Delta\lambda=2$. Thus, we have:

$$W^* = (2W_1, 2W_2, \cdots, 2W_n, 2/C_1, 2/C_2)^T, (C_1 > 0), (C_2 > 0). \quad \text{Q.E.D.}$$

(Necessity) Suppose that after adding two new elements, the judgment matrix A^* is strong rank preserving, and the principal eigenvalue and eigenvector of A^* are given by $\lambda_{max}(A^*)$ and $W^* = (W^*_1, W^*_2, \ldots, W^*_{n+1})^T$, respectively. Similarly, assign $\lambda_{max}(A) = \lambda$, and $\lambda_{max}(A^*) = \lambda + \Delta\lambda$. From (1), (11)-(15) we have:

$$A\tilde{W}^* = \lambda\tilde{W}^* + \triangle\lambda\{\tilde{W}^* - [(a_{1\,n+1}/c_1, a_{2\,n+1}/c_1, \cdots, a_{n\,n+1}/c_1)^T + (a_{1\,n+2}/c_2, a_{2\,n+2}/c_2, \cdots, a_{n\,n+2}/c_2)^T]\} \quad (16)$$

Evidently,

$$\tilde{W}^* = (a_{1\,n+1}/c_1, \ldots, a_{n\,n+1}/c_1)^T + (a_{1\,n+2}/c_2, \ldots, a_{n\,n+2}/c_2)^T$$

satisfies (16). Since A^* is strong rank preserving, we have:

$$\tilde{W}^* = (W_1^*, W_2^*, \cdots, W_n^*)^T = (2W_1, 2W_2, \cdots, 2W_n)^T \quad (17)$$

Comparing (16) and (17), we obtain:

$$a_{i\,n+1} = c_1 W_i \;\; (i=1,2,\ldots,n),\;\; c_1 > 0$$

$$a_{i\,n+2} = c_2 W_i \;\; (i=1,2,\ldots,n),\;\; c_2 > 0$$

Q.E.D.

The result of Corollary 1 can be easily extended to the case in which m new alternatives are added to the set.

Corollary 2. Let the judgments of the upper triangular matrix A of order n be consistent. Let λ_{max} (A) = n and $W = (W_1, W_2, \ldots, W_n)^T$ be the principal eigenvalue and eihenvector of A, respectively. Suppose that the upper triangular matrix of order n+m is given by A^*:

$$A^* = \begin{bmatrix} A & \begin{matrix} a_{1\,n+1} & a_{1\,n+2} & \cdots & a_{1\,n+m} \\ \vdots & \vdots & & \vdots \\ a_{n\,n+1} & a_{n\,n+2} & \cdots & a_{n\,n+m} \end{matrix} \\ & \begin{matrix} n+1 & a_{n+1\,n+2} & \cdots & a_{n+1\,n+m} \\ & \ddots & & \vdots \\ & O & & n+m \end{matrix} \end{bmatrix}, \text{where} \quad A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & 2 & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & O & & n \end{bmatrix}$$

λ_{max} (A^*) and $W^* = (W^*_1, W^*_2, \ldots, W^*_{n+1}, W^*_{n+2}, \ldots, W^*_{n+m})^T$ are the principal eigenvalue and eigenvector of A^*, respectively. If m new elements are added, the necessary and sufficient condition for strong rank preservation is given by:

$$\begin{cases} a_{i\,n+1} = C_1 W_i (C_1 > 0)(i = 1,2,\cdots,n) \\ a_{i\,n+2} = C_2 W_i (C_1 > 0)(i = 1,2,\cdots,n) \\ \vdots \\ a_{i\,n+m} = C_m W_i (C_m > 0)(i = 1,2,\cdots,n) \text{ for the judgement matrix } A \end{cases}$$

3. Weak Rank Preservation

Lemma. For a matrix of incomplete judgments the priorities of the alternatives obtained using GEM [2] are given by:

$$W_i = [1/(n-i)]\left(\sum_{j=+1}^{n} a_{ij} W_j\right), (i = n-1, n-2, \cdots, 1) \tag{18}$$

Theorem 3.1 [3] Let $A = (a_{ij})_{nxn}$ be an upper triangular judgment matrix. After adding a new element, $A^* = (a_{ij})_{n+1xn+1}$. If the elements of the ith row and the kth row satisfy $a_{ij} \geq a_{kj}$ $(j=1,2,\ldots,n)$, and at least there is an inequality which holds, and the new judgment satisfies $a_{i\,n+1} \geq a_{k\,n+1}$, then the relative of importance of the ith element and the kth element does not charge.

Proof: Follows from (18).

Corollary 1 [3]. Under the conditions of Theorem 3.1, if $a_{i\,n+1} = a_{k\,n+1}$, then the relative importance of the ith alternative and the kth alternative does not change.

References

[1] Shubo Xu, The principle of the AHP, page 109, Tiangin University Press, People's Republic of China .

[2] Lianfen Wang, "The Reckoning and Improving of the Rank's Method of Graded Eigenvector Method", Theory and Practice of Systems Engineering, Vol. 9, No. 3, People's Republic of China, 1989.

[3] Lianfin Wang, "New Elements has been Led into the Condition of Rank Preservation in the AHP", Proceedings of Chinese Symposium on the Analytic Hierarchy Process, November 1989, Peoples Republic of China.