A METHOD TO SOVLE MULTIPLE OBJECTIVE DECISION MAKING WITH THE ANALYTIC HIERACHY PROCESS

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ABSTRACT

There are many multiple objective decision making (MODM) in the real world. Every MODM problem has several objectives, sometimes these objectives are contradictory each other. In this paper, a method that converted the multiple objective problem into a single objective problem with weighting method is built, particularly a method is given to calculate weighting coefficients with judgement matrix and its eigenvector and to analyse sensitivity of the matrix with AHP

The MODM problems are pervasive in every field of engineering technology and social economics. mathematically, this kind of problems can be represented as:

$$V\text{——}\min\,[\,f_1(x),\ f_2(x),\cdots f_p(x)] \tag{1}$$

$$R = \{\,x\,|\,g_i(x)\leqslant 0,\ i=1,2,\cdots,m\} \tag{2}$$

where $x = (x_1,x_2\cdots x_n)^T \in$ En. P$\geqslant$2. the notation V-min is different from single objective minimization. The problem consists of n decision variables. m constraints and p objectives. Any or all of the function may be nonlinear. In literature, this problem is often refered to as a vector minimum problem (VMP).

Traditionally there are two approaches for solving the VMP. One of them is to optimize one of the objectives while appending the other objectives to a constraint set. The other approach is to optimize a supper-objective function created by multiplying each objective function with a suitable weight and then by adding them together. This approach leads to the solution of the following problem:

$$\min \sum_{i=1}^{p} w_i f_i(x) \qquad \text{Subject to: } g_i(x)\leqslant 0,\ i=1,2,\cdots,m \tag{3}$$

where $\sum_{i=1}^{p} w_i=1$, $w_i\geqslant 0$, and $w=(w_1,w_2,\cdots,w_p)^T$ is called weighting coefficients. Generally speaking, there is no absolute optimum solution of VMP. Thus. Pareto solution or nondominated solution are defined to overcome this difficulty.

A nondominated solution is one in which no one objective can be improved without a simultaneous detriment to at least one of the other objection of the VMP. That is, x^* is a nondominated solution to the VMP if and only if there doesn't exist x R such that $f_i(x)<f_i(x^*)$ for all i and $f_j(x)<f_j(x^*)$ for at least one j.

We also can prove that the optimum solution of (3) must be the Pareto solution of VMP. Solving problem VMP includes two parts: one is to determine the weighting coefficients with AHP, the another is to optimize linearly weighting problem(3).

EIGENVALUE PROBLEM AND SENSITIVITY ANALYSIS

The process of solving eigenvalue problem can be devided tree steps.

Step1: According to 1-9 ratio scales and to the relative impotance of objectives, the judgement matrix A about $\mathbf{w}=(w_1, w_2,\cdots,w_p)^t$ is established

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{pmatrix} \tag{4}$$

where A is a positive reciprocal matrix, that is, matrix A satisfies the following conditions:

$$a_{ij} > 0 , \quad a_{ij} = \frac{1}{a_{ji}} , \quad a_{ii}=1 \tag{5}$$

Step2: According the Perron theory, every positive matrix must have a maximum eigenvalue and a corresponding eigenvector which whose all components are positive. Let λ, and $\bar{w}$ be the maximum eigenvalue and eigenvector espectively

$$\lambda = \max_{x\in R_p} \phi(x) \tag{6}$$

where

$$R_p^+ = \{x = (x_1,x_2,\cdots, x_n)^T | \, x_i \geq 0, \ x \neq 0\} \tag{7}$$

and

$$\phi(x) = \min x_i^{-1}\sum_{j=1}^{p} a_{ij}x_j \tag{8}$$

Structure linear equation set about $\mathbf{w}$:

$$\begin{bmatrix} a_{11}-\lambda_1 & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22}-\lambda_1 & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pp}-\lambda_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

and obtain its nonzero normalized solution.

Step3: If the decision maker is satisfied with eigenvector $\underline{w}$, then we take this solution as weighting of problem (3), otherwise modify eigenvector with following formula

$$\Delta w = \lim \frac{kA^{k-1}[\|A^k\|(A\cdot E) - \|A^{k-1}(A\cdot E)\| A]e}{\|A^k\|[\|kA^{k-1}(A\cdot E)\| - (k-1)\| A^k\|]} \tag{10}$$

The above process can be repeated.

SINGLE OBJECTIVE OPTIMIZATION

Consider weighting problem (3) which is a nonlinear programming with inequality constrains. We take the Zoutendijk feasible direction method solving problem(3). Given a feasible point x_k of problem(3), a direction d_k is determined such that for $\lambda > 0$ and sufficiently small, the following two properties are true: (1) $x_k + \lambda d_k$ is feasible. and(2) the objective value at $x_k + \lambda d_k$ is better than the objective value at x_k. After such a direction is determined, a one-dimensional optimization problem is solved in order to determine how far to proceed along d_k. This leads to a new point x_{k+1}, and the process is repeated. We now consider the constrainted nonlinear programming (3), and rewrite it in the following form:

$$\text{minimize } H(x) \tag{11}$$

$$\text{Subject to } g_i(x) \leq 0, \quad i=1,2,\cdots, m. \tag{12}$$

where

$$H(x) = \sum_{i=1}^{p} w_i f_i(x) \tag{13}$$

Let x be a feasible solution, and let I be the set of binding constraints, that is, $I = \{ i | g_i(x)=0 \}$. Furthermore, suppose that f and g_i for $i \in I$ are differentiable at x and g_i for $i \notin I$ is continuous at x. If $f(x)^t d < 0$ and $g_i(x)^t d < 0$, $i \in I$, then d is an improving feasible direction. In order to find a vector d satisfying $f(x)^t d < 0$ and $g_i(x)^t d < 0$ for $i \in I$, it is only natural to minimize the maximum of $f(x)^t d$, and $g_i(x)^t d$ for $i \in I$. Denoting this maximum by z, we get the folowing direction-finding problem:

$$\text{minimize } z \tag{14}$$

$$\text{Subject to} \quad f(x)^t d - z \leqslant 0 \tag{15}$$

$$g_i(x)^t d - z \leqslant 0 \quad \text{for } i \quad I \tag{16}$$

$$-1 \leqslant d_j \leqslant 1 \quad \text{for } j=1,2,\cdots,n \tag{17}$$

let (z,d) be an optimal solution to the above linear problem. If $z<0$, then d is obviously an improving feasible direction. If, on the other hand, $z=0$, then the current vector is a Fritz John point.

CONVERGENCE OF THE METHOD

We now prove that the opotimal solution of weighting problem (3) is a Pareto solution.

Theorem 1 Suppose that $w_i>0$, $i=1,2,\cdots p$, $\sum_{i=1}^{p} w_i=1$, then $h(f(x)) = \sum_{i=1}^{p} w_i f_i(x)$ is a strict monotonous function of $F(x)=(f_1(x),f_2(x),\cdots,f_p(x))^T$.

Proof

Because of $w_i>0$, $i=1,2,\cdots,p$, and $f_i<\bar{f}_i$, then there exist at least one $i_0(1<i_0<p)$ such that

$$f_{i_0} < \bar{f}_{i_0} \tag{18}$$

therefore

$$w_{i_0} f_{i_0} < w_{i_0} \bar{f}_{i_0} \tag{19}$$

and

$$w_i f_i \leqslant w_i \bar{f}_i, \quad i=1,2,\cdots p \tag{20}$$

thus

$$h(F) = \sum_{i=1}^{p} w_i f_i < \sum_{i=1}^{p} w_i \bar{f}_i = h(\bar{F}) \tag{21}$$

and the proof is complete.

Theorem 2 Let $h(f(x))$ be a strict monotonous function of F, them the optimal solution x of single optimization problem

$$\min h(F(x)) = \min \sum_{i=1}^{p} w_i f_i(x) \tag{22}$$

is a Pareto solution of multiple objective decision making VMP

Proof

Suppose, by contradiction, that x isn't a Pareto solution, that is, these exists $y \in R^n$ such that

$$F(y) < F(x) \tag{23}$$

because of the strict monotonicity, we have

$$h(F(y)) < h(F(x))$$

The relationship above contradicts that x is a optimal solution of min $h(F(x))$. Therefore x is a Pareto solution of VMP, and the proof is complete.

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