IMPROVING THE ORDINAL INCONSISTENCY OF PAIRWISE COMPARISON MATRICES

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ABSTRACT

This paper investigates the effects of non-transitive judgments on the consistency of pairwise comparison matrices and proposes a heuristic algorithm for identifying and eliminating their ordinal inconsistencies. The algorithm is based on graphical representation of the comparison matrices and identifies the edges in the digraph, which are mostly responsible for three-way cycles, representing the ordinal inconsistencies. The algorithm tries to minimize the number of edge reversals and provides results, similar to those obtained by an optimization method.

Keywords: Pairwise Comparisons, Inconsistency, Non-transitive Judgements, Decision Analysis, Heuristics

1. Introduction

Pairwise comparisons are a vital part of the prioritisation procedure in the Analytic Hierarchy Process (AHP), which provides a comprehensive and rational framework for structuring a decision problem (Saaty, 1980). In the AHP, the pairwise judgements are structured in pairwise comparison matrices (PCM), and some prioritisation procedures are applied to derive a corresponding priority vector. If the comparison judgements are cardinally consistent, then the PCM are also consistent, and all prioritisation methods give the same result. However, in the case of ordinal or cardinal inconsistent judgements, different prioritisation methods derive different priority vectors.

The AHP does not require transitivity of the comparison judgements. Saaty's CR index measures the cardinal inconsistency of the judgements, but does not capture the ordinal inconsistency. Ordinal inconsistency always implies cardinal inconsistency, however, the converse does not hold. Generally, if the comparison judgements and the corresponding PCM are ordinally inconsistent, the level of their cardinal inconsistency is considerably high; therefore the AHP implicitly presumes that satisfying the CR test may significantly reduce the chances of ordinal inconsistency. However, there are examples in the literature where matrices that satisfy the CR criterion can also be ordinarily inconsistent (Jensen and Hicks, 1993; Kwiesielewicz and van Uden, 2004).

The removal of intransitivities can be formulated as a non-linear integer programming problem problem (Mikhailov et al, 2010). However, its solution requires applying a complex optimisation procedure and is

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rather difficult from computational point of view. This paper proposes a heuristic procedure for improving the overall consistency of PCM by detecting and modifying inconsistent ordinal judgements. The heuristic algorithm achieves almost identical results to the optimisation algorithm; however, it does not require applying numerical methods and is much more efficient from a computational viewpoint.

2. Pairwise Comparison Matrices

Consider a prioritization of *n* elements $E_1, E_2, ..., E_n$ at the same level hierarchy. The decision maker (DM) assesses the relative importance of any two elements E_i and E_j by providing a comparison judgment a_{ij}, specifying by how much E_i is preferred/not preferred to E_j. The AHP method structures any set of comparison judgements in a positive reciprocal PCM, $A = [a_{ij}]$. Then a priority vector $w = (w_1, w_2, ..., w_n)^T$ could be obtained from *A*, by applying some prioritization method (Choo and Wedley, 2004).

2.1 Consistency of PCM

The judgments of DMs are *cardinally consistent* if $a_{ij} = a_{ik} * a_{kj}$ for all *i* and *j*; where $j > k > i$ (Saaty 1980). When the judgments of the DM are perfectly consistent, then the judgements a_{ij} have perfect values $a_{ij} = w_i / w_j$. In such a case, the PCM is said to be (perfectly) *consistent*. If the DM's judgements are *cardinally inconsistent* (i.e. $a_{ij} \neq a_{ik} * a_{kj}$ for some *i, j, k*) then the corresponding comparison matrix *A* is said to be *inconsistent* and we have $a_{ij} \approx w_i / w_j$.

Saaty (1980) proposed a measure of consistency based on the properties of positive reciprocal matrices and defines a measure of consistency, called a Consistency Ratio (*CR*). If the value of *CR* is smaller than or equal to 0.1, the estimated priority vector w can adequately approximate the unknown preference vector r, therefore, the PCM is of *acceptable* inconsistency. However, if *CR*>0.1, the estimated priorities could be erroneous and the DMs should be asked to improve the consistency by revising their subjective judgments.

The *ordinal consistency*, which is also known as a transitivity condition between 3 elements, states that if E_i is preferred to E_j and E_j is preferred to E_k, then E_i should be preferred to E_k. Using the preference symbol $\rightarrow$, the ordinal consistency is represented as: if $E_i \rightarrow E_j \rightarrow E_k$ then $E_i \rightarrow E_k$. If however, $E_k \rightarrow E_i$ when $E_i \rightarrow E_j \rightarrow E_k$, then the preference judgements are *ordinally inconsistent*, or intransitive. Therefore, the ordinal inconsistency could be defined as $E_i \rightarrow E_j \rightarrow E_k \rightarrow E_i$, which represents a *circular triad* of preferences (Kendall and Smith 1940). In terms of PCM, the preference relation $E_i \rightarrow E_j$ means that $a_{ij} > 1$. Therefore, the ordinal inconsistency $E_i \rightarrow E_j \rightarrow E_k \rightarrow E_i$ means that the corresponding judgements are $a_{ij} > 1$, $a_{jk} > 1$, $a_{ki} > 1$.

2.2 Graphical representation of PCM

The relationships between the elements of a PCM in the case of preference dominance can be depicted by a *directed graph* (digraph), where the intransitive relationship $E_i \rightarrow E_j \rightarrow E_k \rightarrow E_i$ is represented as a *three-way cycle* between the three elements (Kendall, 1955).

Consider a problem with 5 comparison elements, where the DM provides the following PCM:

$$A=\begin{array}{c} \begin{matrix} E_1 & E_2 & E_3 & E_4 & E_5 \end{matrix} \\ \begin{bmatrix} 1 & 7/4 & 3/4 & 5/2 & 7/4 \\ 4/7 & 1 & 3/4 & 9/4 & 9/4 \\ 4/3 & 4/3 & 1 & 3/4 & 3/4 \\ 2/5 & 4/9 & 4/3 & 1 & 5/8 \\ 4/7 & 4/9 & 4/3 & 8/5 & 1 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{matrix} \end{array} \tag{1}$$

In the digraph of this matrix, shown in Fig. 1, each comparison element E_i is represented as a node and the judgements as edges between the nodes. For example, the directed edge from E_1 to E_2 shows that E_1 is preferred to E_2, whereas the weight of the edge represents the intensity of the preference, which is equal to the value of the pairwise comparison judgement between these two elements, $a_{12}=2$. It can be seen that the diagraph contains four three-way cycles: $E_1 \to E_4 \to E_3 \to E_1$, $E_1 \to E_5 \to E_3 \to E_1$, $E_2 \to E_4 \to E_3 \to E_2$ and $E_2 \to E_5 \to E_3 \to E_2$. As the comparison judgements are intransitive and there are cycles in its digraph, the comparison matrix (1) is ordinally inconsistent.

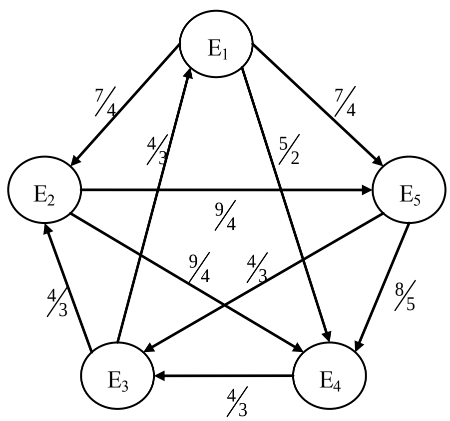

Fig. 1. Preference graph with cyclic judgments

In PCMs of dimension *n*>3, cycles with more than three ways may exist. For example, in the graph shown in Fig. 1, there is a four-way cycle $E_1 \to E_5 \to E_4 \to E_3 \to E_1$. However, it is important to note that if there are no three-way cycles in the digraph, then there are no cycles of higher order (Gass, 1998).

2.3. Priority violations

When $E_i \to E_j$, it is assumed that the priorities of the elements should preserve the preference direction, i.e. $w_i > w_j$. However, if E_j receives a larger priority weight, $w_j > w_i$, then a *priority violation* occurs (Ali et al. 1986).

The number of violations (*NV*) is defined as follows:

$$NV(w)=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} v_{ij} , \tag{2}$$

$$v_{ij} = \begin{cases} 1, & if\ w_i > w_j\ and\ a_{ij} < 1, or\ w_i < w_j\ and\ a_{ij} > 1, \\ 1/2, & if\ w_i = w_j\ and\ a_{ij} \neq 1, or\ w_i \neq w_j\ and\ a_{ij} = 1, \\ 0, & otherwise. \end{cases}$$

If the PCM is ordinally inconsistent, it is not possible to find a priority vector that satisfies all preference directions expressed by the ordinal comparisons and, therefore, there will always be priority violations.

The priority vectors *w* for the PCM of this example, obtained by the Eigenvector method (EV), the Direct Least Squares (DLS), the Logarithmic Least Squares (LLS), the Logarithmic Least Absolute Value (LLAV) and the Logarithmic Absolute Error (LAE) (Choo and Wedley, 2004) are shown in Table 1.

Table 1. Priorities derived by different prioritisation methods

Method	w1	w2	w3	w4	w5	*NV*
EV	0.269	0.226	0.203	0.134	0.169	4
DLS	0.282	0.239	0.191	0.124	0.164	4
LLS	0.275	0.227	0.194	0.132	0.172	4
LLAV	0.303	0.253	0.139	0.120	0.186	3
LAE	0.283	0.255	0.187	0.113	0.162	4
MNV	0.350	0.286	0.086	0.136	0.143	2

From the results it can be seen that the EV, the DLS and LLS methods derive priority vectors with the same ranking order $w_1 > w_2 > w_3 > w_5 > w_4$ (but with different intensities). This ranking has four violations, i.e. $w_1 > w_3$, $w_2 > w_3$, $w_3 > w_4$ and $w_3 > w_5$ (as $a_{13} < 1$, $a_{23} < 1$, $a_{34} < 1$, $a_{35} < 1$). The LAE method generates a priority vector with a different rank order, which, however, also has four violations, whereas the LLAV method produces a ranking with only three violations. The priority vector with Minimum *NV* (MNV) is obtained by applying an optimization procedure for minimizing (2). No prioritization method can obtain a vector with less than 2 priority violations.

It should be noted that the PCM (1) used in this example is of acceptable inconsistency, as its *CR*=0.083. Hence, by applying the *CR* test only, the PCM will be classified as acceptable and the DM will not be asked to reconsider their judgements, therefore, regardless of the used prioritisation method, erroneous results will be obtained. It is obvious that the ordinal inconsistency is the main cause for obtaining priorities, which do not correspond to the DM's preferences. Therefore, the improvement of ordinal inconsistency by elimination of intransitivities could be regarded as the most important way to increase the accuracy of the decision-making process

3. Heuristic Approach for Rectification of Intransitive Judgments

Elimination of intransitive judgements can be formulated as an optimization problem to remove cycles by changing a minimal number of elements in the PCM (Mikhailov, 2010). However, in the case of pairwise comparison judgments, the available cardinal information can be additionally used to further sift the possible edges to be reversed.

Consider a partial preference digraph between any two elements E_i and E_j. Let α be the number of times that E_i is preferred over other elements in the overall digraph. Similarly, let β be the number of times when E_j is preferred over other elements. Fig. 2 shows a digraph, where E_i is preferred to E_j as the edge v_{ij} is directed towards E_j. The element E_i is also preferred to four other elements, therefore, its number of outflows is α =5. Similarly, E_j is preferred to three other elements, so it has three outflows. Kendall and Smith (1940) showed that when $\alpha > \beta$ and the edge v_{ij} is reversed, the number of three-way cycles in the overall preference digraph is increased. Moreover, when $\alpha < \beta$, reversing the direction of the edge v_{ij} reduces the number of three-way cycles. This observation can be used to develop an iterative Heuristic algorithm, which can reduce both the intransitive elements and the number of three-way cycles.

At each iteration of the proposed Heuristic algorithm, the difference between the outflows of any two elements E_i and E_j is calculated and the edge, satisfying the condition $\max_{i,j}(\beta_j - \alpha_i)$ is reversed. In the case of multiple edges meeting this condition, the most inconsistent edge is to be reversed, using the cardinal consistency criterion $a_{ij} = a_{ik}a_{kj}$.

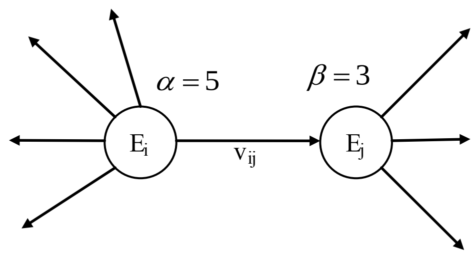

Fig. 2. Outflows for source and destination elements

Let γ_{ij} be a value measuring the level of inconsistency contributed by v_{ij} to the overall inconsistency of the pair-wise comparisons. The value of γ_{ij} is calculated as the mean logarithmic deviation for all indirect judgments $a_{ik}a_{kj}$ from the direct judgement a_{ij}, i.e.

$$\gamma_{ij} = \frac{1}{(n-2)}\sum_{k=1}^{n}(\log(a_{ij}) - \log(a_{ik}a_{kj})), \text{ where } k \neq i \neq j.$$

Consider again the digraph of the 5-dimensional PCM, shown on Fig. 1. The values of α and β for all its edges are listed in Table 2.

Table 2. The values of *α, β* and *γ* for all edges of the digraph in Fig. 1 at the first iteration

v	*α*	*β*	*β - α*	*γ*
v_{12}	3	2	-1	0.61
v_{14}	3	1	-2	0.69
v_{15}	3	2	-1	0.69
v_{24}	2	1	-1	0.77
v_{25}	2	2	0	0.89
v_{31}	2	3	1	1.06
v_{32}	2	2	0	1.11
v_{43}	1	2	1	1.12
v_{53}	2	2	0	1.00
v_{54}	2	1	-1	0.35

According to the algorithm, the edges with the maximum value of $\beta-\alpha$ should be identified. In this case, v_{31} and v_{43} are the edges with maximum value of $\beta-\alpha$ =1. Therefore, the values of γ_{31} and γ_{43}, shown in Table 2 are used to select the most inconsistent edge. As the level of inconsistency γ_{43} is greater than γ_{31}, the edge v_{43} should be reversed. In terms of the elements of the initial comparison matrix (1), we interchange the values of the corresponding comparison elements a_{43} and a_{34}, so the updated values are a_{43}=3/4 and a_{34}=4/3, respectively. After the first iteration, two of the four cycles are removed from the original PCM. The two cycles remaining in the updated PCM are $E_1 \rightarrow E_5 \rightarrow E_3 \rightarrow E_1$ and $E_2 \rightarrow E_5 \rightarrow E_3 \rightarrow E_2$.

In the next iteration, the edges with the maximum value of $\beta-\alpha$ are identified for the updated PCM. The edge v_{53} has the maximum value of $\beta-\alpha$ equal to 1. The value of γ is irrelevant in this iteration, as there are no other edges with such $\beta-\alpha$. Therefore, the comparison elements a_{53} and a_{35} are swapped and their new values become a_{53}=3/4 and a_{35}=4/3, respectively. Thus, the updated PCM has no three-way cycles and becomes transitive. Its *CR* is improved to 0.055 from the original value of 0.083. It can be shown that the obtained heuristic solution is equivalent to optimal solution to this problem, however, it is both much faster and simpler from a computation point of view.

4. Conclusions

This paper justifies the need to improve the ordinal consistency and proposes an efficient heuristic algorithm to eliminate the intransitivity of PCMs. The proposed approach does not replace the CR consistency test. It could be considered as an additional procedure for improving the overall consistency of the comparison judgements, and consequently, the quality of the decision making process.

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