AHP and Two-Envelope Paradox Resolution via Nonlinear Scaling

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- Two Envelopes paradox presents a fascinating problem in probability and decision making, it maybe is one of the most famous problems not yet totally resolved.
- The player is presented with two envelopes and informed that one of them contains twice as much money as the other one. The player takes one of them without looking inside, and is then given the opportunity to change their mind and take the second envelope instead of the first one.
- Assuming that the 1st envelope contains a value A, then the 2nd one can have 2A or A/2 with equal probability, and its expected value is the mean 1.25A, or 25% profit from switching envelopes.
- It is a great result, but we could denote the amount in the 2nd envelope as A and repeat the derivation outcome 1.25A already in 1st envelope, so each of them is worth more than the other one.
- So, to switch or not to switch? – That is the question.
- It has been studied in multiple works based on probability theory and decision making, Bayesian evaluations, game theory and mathematical economics, logic and epistemology.
- An extensive description of its different versions and trying to resolve the problem can be found on the web (see Wikipedia).

- **Several examples of references:**

- Abbott D., Davis B.R., and Parrondo J.M. (2010) The two-envelope problem revisited, Fluctuation and Noise Letters, 9, 1-8.
- Albers C.J., Kooi B.P., and Schaafsma W. (2005) Trying to resolve the two-envelope problem, Synthese, 145, 89–109.
- Chalmers D.J. (2002) The St. Petersburg Two-Envelope Paradox, Analysis, 62, 2, 155-157.
- Egozcue M. and Garcia L.F. (2015) An optimal threshold strategy in the two-envelope problem with partial information, J. of Applied Probability, 52, 1, 298-304.
- Falk R. and Nickerson R. (2009) An inside look at the two envelopes paradox, Teaching Statistics, 31, 39-41.
- **Kraitchik M.B. (1953)** Mathematical recreations, Dover, New York, NY. – **First formulation!**
- Lee Ch. (2013) The Two-envelope Paradox: Asymmetric Cases, Mind, 122, 485, 1-26.
- Linzer E. (1994) The Two Envelope Paradox, The American Mathematical Monthly, 101, 5, 417-419.
- Markosian N. (2011) A Simple Solution to the Two Envelope Problem, Logos & Episteme, II, 3, 347–357.
- McDonnell M.D., Grant A.J, Land I., Vellambi B.N., Abbott D., and Lever K. (2011) Gain from the two-envelope problem via information asymmetry: on the suboptimality of randomized switching". Proceedings of the Royal Society A, 467, 2825–2851.
- Nalebuff B. (1989) The other person´s envelope is always greener, J. of Economic Perspectives 3, 171-81.
- Schwitzgebel E. and Dever J. (2008) The two-envelope and using variables within the expectation formula, Sorites, 20, 135–140.
- Syverson P. (2010) Opening Two Envelopes, Acta Analytica 25, 4, 479–498.
- Yi B.U. (2013) Conditionals and a Two-envelope Paradox, The Journal of Philosophy, 110, 5, 233-257.

- In a description of the paradox's main version as a non-informed game, the player is presented with two envelopes and told that one of them contains twice as much money as the other envelope.
- The player takes one of them without looking inside, and then has the option to change their mind and take another envelope instead of the first one. It seems that without any information of the content the player can value both envelopes equally, so there is no reason to substitute one with another.
- However, assuming that the 1st envelope contains a value A, then the 2nd can have $2A$ or $A/2$ equally probable, and its expected value is the mean value

$$(2A+0.5A)/2=1.25A \, , \qquad (1)$$

which is 25% profit from switching envelopes.

But we could denote the amount in the 2nd envelope as A and repeat the derivation getting $1.25A$ already in 1st envelope, so each of them is worth more than the other one!

The grass is always greener on the other side, so to switch or not to switch?

- As shown in this work, the ideas borrowed from the Analytic Hierarchy Process help in resolving this paradox by transforming the ratio scale into the additive or logarithmic scales which correspond to application of the multiplicative utility function.
- The reason for it is as follows:
- Measuring values in the second envelope are two times larger or smaller than in the first envelope, actually corresponds to the ratio scale.
- However, a ratio scale is not an additive, and if an addition is performed in the ratio scale it could easily produce unclear or strange results, exactly as it happens in the paradox of two envelopes.
- Resolution of this problem can be achieved by transforming the ratio scale into a scale which permits the operation of addition.
- Such a transformation can be done with the share and logarithm of the quotients – in nonlinear scales the expected value equals zero, and the corresponding expected value in the original scale equals one, thus, there is no gain in changing envelopes, and the paradox is dissolved.
- Let us describe these scales in more detail.

Transformation of mutually reciprocal values into their shares

- The ratio scales are well-known in multiple-criteria decision making methods, for example, the AHP widely used for solving problems of prioritization is based on the data elicited via pairwise quotients.
- Due to AHP founder Thomas Saaty, the pairwise comparisons among multiple criteria or alternatives are made in the ratio scale of how many times one criterion is preferred over another one. After eliciting the pairwise ratios from an expert, the matrix of the pairwise ratios is built, with the elements a_{ij} of ratio of priorities of the i-th to j-th items.
- If one criterion is K times preferred over another criterion then the ratio of their preferences equals K. The value K is measured by the natural numbers from 1 when the criteria are equal, 2 for a slight preference, etc., up to 9 of the overwhelming preference.
- If a preferences' ratio of the i-th to j-th item equals K then the opposite preferences' ratio of the j-th to i-th item equals the reciprocal value $1/K$. The matrix elements are reciprocally transposed, so $a_{ij} =1/ a_{ji}$, for instance, if $a_{ij} = 2$ then $a_{ji} = 0.5$, or if $a_{ij} = 8$ then $a_{ji} = 0.125$.
- The bigger values have a bigger impact in numerical estimations performed in the regular additive scale because of the unbalanced impact of the bigger ratios and their too small reciprocal counterparts.
- The classical AHP takes it into account in solution via eigenproblem of such a matrix, with its first eigenvector serving as the vector of the items' priorities.

- In the works (Lipovetsky S. and Conklin M., Robust Estimation of Priorities in the AHP, European J. of Operational Research, 137, 2002, 110-122; Lipovetsky S. and Conklin M., AHP Priorities and Markov-Chapman-Kolmogorov Steady-States Probabilities, International J. of the Analytic Hierarchy Process, 7 (2), 2015, 349-363), it was suggested to transform a matrix of pairwise ratios into the matrix of their shares.
- Using the idea of reciprocal numbers' conversion into the balanced additive scale for the problem of two envelopes, let us consider the transformation of a value *K* and 1/*K* into their shares S_K and $S_{1/K}$, respectively. It can be done by the following formulae:
- $$S_K = \frac{1}{1+\frac{1}{K}} = \frac{K}{K+1}; \quad S_{1/K} = \frac{1}{1+\frac{1}{1/K}} = \frac{1}{K+1}; \quad S_K + S_{1/K} = 1 . \qquad (2)$$
- For an example of *K*=2 and 1/*K*=0.5, the shares of these values by (2) are:
- $$S_{K=2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}; \quad S_{\frac{1}{K}=0.5} = \frac{1}{1+\frac{1}{1/0.5}} = \frac{1}{3}; \quad S_2 + S_{0.5} = 1 . \qquad (3)$$
- For *K*=1, as in the first original envelope's value, the transformation (2) yields the share
- $$S_{K=1} = \frac{1}{1+\frac{1}{1}} = \frac{1}{2} . \qquad (4)$$

- The deviations D of the share values (2) from the basic level 1/2 are as follows:

$$DS_K = \frac{K}{K+1} - \frac{1}{2} = \frac{K-1}{2(K+1)}; \quad DS_{\frac{1}{K}} = \frac{1}{K+1} - \frac{1}{2} = \frac{1-K}{2(K+1)}; \quad DS_K + DS_{1/K} = 0, \qquad (5)$$

so the centered shares' total equals zero.

- For the example with K=2, the deviations of the shares from the basic level calculated by (5) are:

$$DS_{K=2} = \frac{2-1}{2(2+1)} = \frac{1}{6}; \quad DS_{\frac{1}{K}=0.5} = \frac{1-2}{2(2+1)} = -\frac{1}{6}; \quad DS_{K=2} + DS_{\frac{1}{K}=0.5} = 0\,. \qquad (6)$$

The expected utility with equal probabilities, or the mean value of the shares calculated by the nonlinear scale is:

$$0.5\, S_2 + 0.5 S_{0.5} = \frac{1}{2}\,. \qquad (7)$$

It equals the basic level and coincides with the utility in the first envelope, thus, no substitution of the envelopes changes this utility. Therefore, in the adequate scale of shares for measuring utility, the paradox dissolves.

Multiplicative utility and logarithmic transformation

- The ratio scales are transformed into the logarithmic scale in the multiplicative AHP:

- Saaty T.L. and Vargas L.G., Comparison of eigenvalue, logarithmic least squares and least squares methods in estimating ratios, Mathematical Modelling, 5, 1984, 309-324;

- Saaty T.L. and Vargas L.G., Decision Making in Economic, Political, Social and Technological Environment with the Analytic Hierarchy Process. RWS Publications, Pittsburgh, PA, 1994;

- Lootsma F., Scale sensitivity in the multiplicative AHP and SMART, Journal of Multi-Criteria Decision Analysis, 2, 1993, 87-110;

- Lootsma F., Multi-Criteria Decision Analysis via Ratio and Difference Judgement. Kluwer Academic Publishers, Dordrecht, 1999;

- Lipovetsky S. and Tishler A., Interval estimation of priorities in the AHP, European Journal of Operational Research, 114, 1999, 153– 164;

- Lipovetsky S., Global Priority Estimation in Multiperson Decision Making, Journal of Optimization Theory and Applications, 140, 2009, 77-91.

- There the pairwise ratios are transformed by logarithm, so the reciprocal values K and $1/K$ become opposite numbers:

$$\log(K) = -\log(1/K)\ . \qquad (8)$$

- These logarithms are equal by their absolute value, so they are comparable in the additive scale.

- Applications of additive and multiplicative modes and their comparison have been studied in various problems, for instance:
- Keeney R.L., Multiplicative Utility Functions, Operations Research, 22, 1, 1974, 22-34;
- Mehrez A., Yuan Y., and Gafni A., Stable solutions vs. multiplicative utility solutions for the assignment problem, Operations Research Letters, 7, 3, 1988, 131-139.

- For K=2 and 1/K = 0.5, which corresponds to the two-envelopes problem, the expected utility with equal probabilities, or the mean value in the logarithmic scale equals:

$$0.5\, log(2) + 0.5 log(\frac{1}{2}) = 0\,. \tag{9}$$

- As log(1)=0, the expected utility in the original scale equals 1, which is the utility in the first envelope with the amount A (1).
- The same conclusion can be obtained if to estimate the expected utility not by the arithmetic mean but by the geometric mean which is an adequate measure for the multiplicative utility.
- The expected value for the multiplicative utility is defined by the weighted geometric mean, so in the two-envelope problem with equal probability of choice between the envelope with 2A and with 0.5A values it is:
- $$(2A)^{0.5}(0.5A)^{0.5} = \sqrt{(2A)(0.5A)} = A\,, \tag{10}$$
- which coincides with the value in the original envelope.
- Thus, no substitution of the envelope changes the original utility, and in the adequate logarithmic scale for measuring utility in the multiplicative mode the paradox is resolved.

- Comparing the shares and logarithmic scale we see they behave very similarly and differ by a constant term depending on the base of logarithm.
- Table 1: share and logarithmic transformations.

ratio K	Shares (5)	Logarithm (8)	log/shares	log-shares	abs(log-shares)
1/9	-0.400	-0.477	1.193	-0.077	0.077
1/8	-0.389	-0.452	1.161	-0.063	0.063
1/7	-0.375	-0.423	1.127	-0.048	0.048
1/6	-0.357	-0.389	1.089	-0.032	0.032
1/5	-0.333	-0.349	1.048	-0.016	0.016
1/4	-0.300	-0.301	1.003	-0.001	0.001
1/3	-0.250	-0.239	0.954	0.011	0.011
1/2	-0.167	-0.151	0.903	0.016	0.016
1	0.000	0.000		0.000	0.000
2	0.167	0.151	0.903	-0.016	0.016
3	0.250	0.239	0.954	-0.011	0.011
4	0.300	0.301	1.003	0.001	0.001
5	0.333	0.349	1.048	0.016	0.016
6	0.357	0.389	1.089	0.032	0.032
7	0.375	0.423	1.127	0.048	0.048
8	0.389	0.452	1.161	0.063	0.063
9	0.400	0.477	1.193	0.077	0.077
mean	**0.000**	**0.000**	**1.060**	**0.000**	**0.031**
std	**0.330**	**0.363**	**0.098**	**0.042**	**0.026**

- In the two bottom rows, there are the mean values and standard deviations (std) in each column, and all of those are very small, so the results are mostly the same by both scales.
- The results are similar for a larger span of the ratios up to 100 as well, with the corresponding reciprocal values.
- For a particular *K*, it is possible to choose such a base *x* for the logarithmic transformation that the centered share value DS_K (5) will be exactly equal to this logarithm (8):

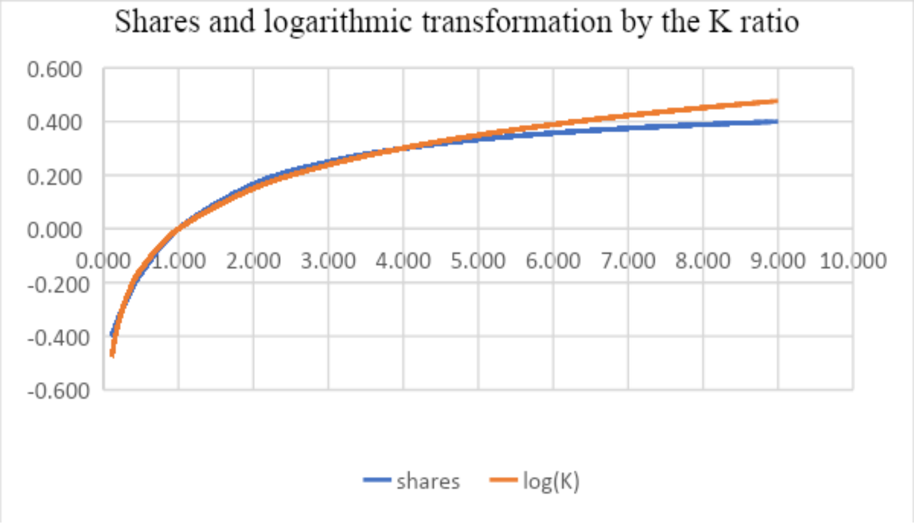
$$log_x K = DS_K \ . \qquad (11)$$

- For example, if *K*=2 as in the two-envelope problem, the centered share value equals 1/6 (as in the formula (6) and in Table 1), and solving the equation (11) yields:

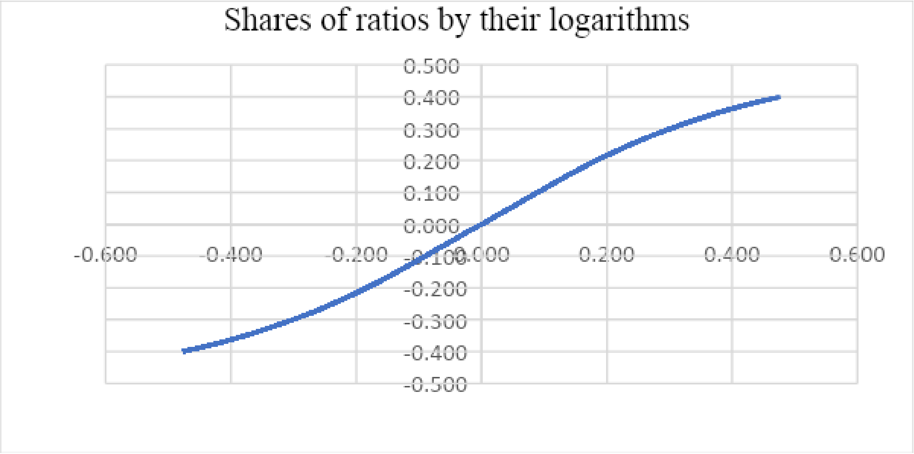
$$x = K^{1/DS_K} = 2^6 = 64 \ . \qquad (12)$$

- Due to (11) change of the ratio *K*=2 by the logarithm of the base 64 gives the same centered share 1/6=0.167, leading to both scaling transformations coinciding by their altered values.

- Shares and logarithm values from Table 1 presented in the graphs.
- Figure 1. Shares and logarithm transformation profiled by the ratio values.



- Figure 2. Shares by logarithms of the ratio values.



- **Summary**
- This work considers the two-envelope paradox and proposes to describe it in the nonlinear scales of the shares or logarithms of the values.
- Such kind of rescaling is known in multiple-criteria decision making methods, specifically, in the Analytic Hierarchy Process.
- It is shown that the two-envelope paradox can be easily dissolved if to use these scales, particularly, the logarithmic scale corresponding to the multiplicative utility with the geometric mean for the expected utility estimation.
- The reason for it is as follows: if in comparison with the original envelope value A, the second envelope contains twice more or twice less, 2A or A/2, then it means that the measurement is performed by the ratio scale with terms of multiplication 2 and 0.5. But the ratio scale is not an additive scale, and if to use the arithmetic mean for the expected utility the result will be biased to the bigger value 1.25A.
- However, if to use the geometric mean, adequate to the ratio scale, then the result coincides with the original value A, and there is no paradox.
- Similar results are achieved with the scale of the shares of the multiplicative terms.

- Numerical comparison shows that both the shares and logarithmic scales are very close in a wide range of possible values of the ratios.
- The fact that nonlinear scaling dissolves the weird conclusion of this paradox that each envelope is better to change to another one proves the correctness of using the nonlinear utility approach.
- This technique helps to understand the two-envelope problem, solves its paradox, and can be useful in other applications of the multiple-criteria decision making.

Thanks!