ON THE SIMILARITY AMONG PRIORITY DERIVING METHODS FOR THE AHP

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THE OBJECTIVE OF THE RESEARCH

- **TO ESTABLISH A RELATION BETWEEN 'GENUINE' AND ESTIMATED PRIORITY VECTORS DERIVED FROM SIMULATED PAIRWISE JUDGMENTS**

- **GOOD OR POOR RELATION WILL REFLECT QUALITY OF SELECTED ESTIMATION TECHNIQUES**

ASSUMPTIONS

GOOD OR POOR RELATIONS ARE INDICATED BY:

Mean Spearman Rank Correlation Coefficient (MSRC)
Mean Pearson Correlation Coefficient (MPCC)
Mean Average Absolute Deviation (MAAD)
average Root Mean Squared Deviation (aRMSD)

$$MAAD(w,\hat{w}) = \frac{1}{n}\sum_{i=1}^{n}\left|w_i - \hat{w}_i\right| \qquad aRMSD(w,\hat{w}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(w_i - \hat{w}_i\right)^2}$$

where w and $\hat{w}$ denote the 'true' priority vector and its estimate, consecutively.

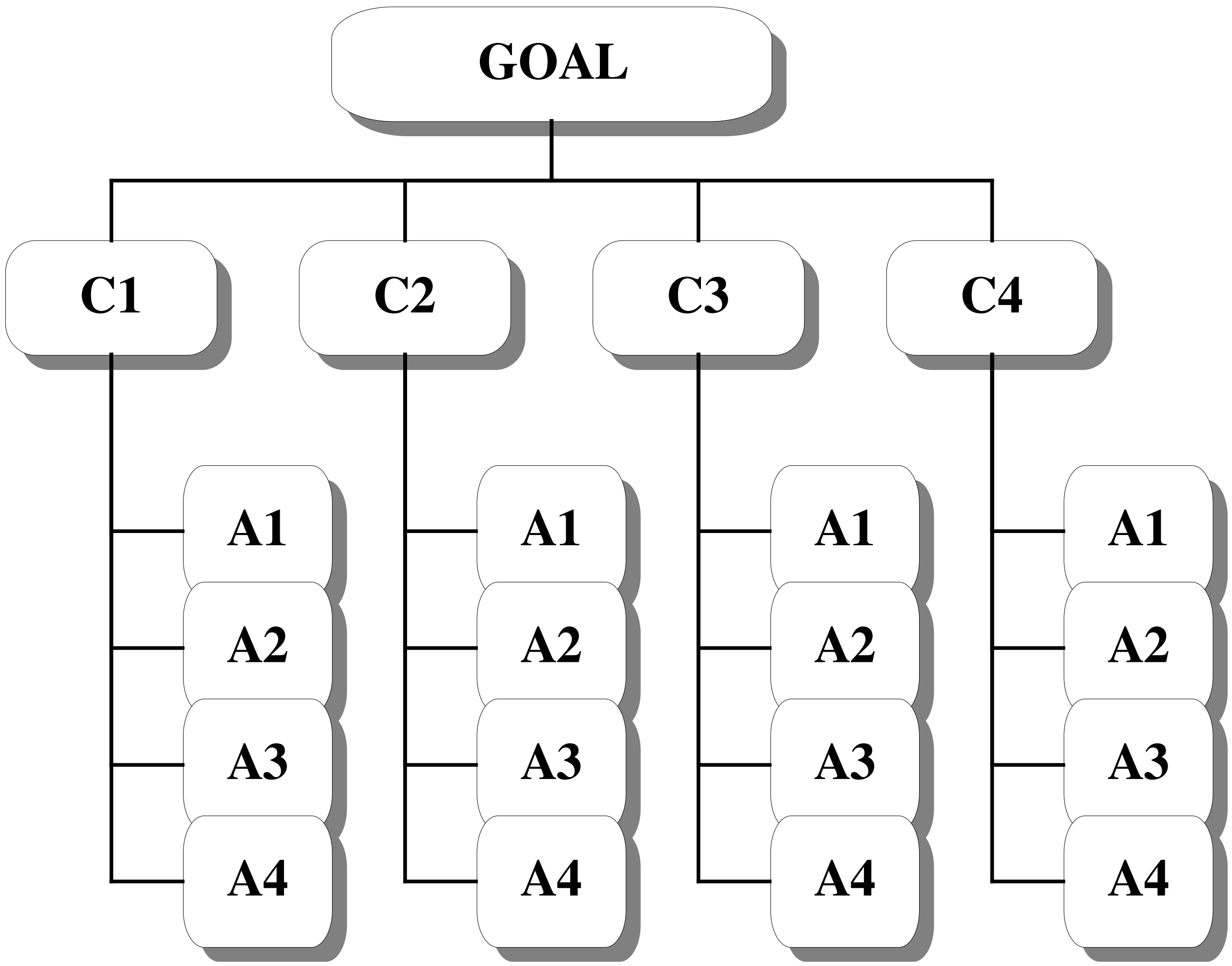
SIMULATION CONCEPT

In order to clarify the examination method we are presenting the simplified example now.

For the illustration purpose we take into consideration only technical perturbation of PCMs resulting from rounding errors during application of Saaty's scale and standard requirement within AHP i.e. forced reciprocity.

Let's consider the following hypothetical model of the AHP framework with three levels (four criteria and four alternatives):

SIMULATION CONCEPT



– PRELIMINARIES –

PAIRWISE COMPARISONS SCALES

Intensity of Importance	Definition
1	Equal Importance
3	Moderate Importance
5	Strong Importance
7	Very Strong Importance
9	Extreme Importance
2, 4, 6, 8	For compromises between the above
Reciprocals of above	In comparing elements i and j - if i is 3 compared to j - then j is 1/3 compared to i
Rationals	Force consistency Measured values available

– PRELIMINARIES –

PAIRWISE COMPARISONS SCALES

Other numerical scales also exist for example various types of geometric scales.

Their most popular version consists of the numbers computed in accordance with the formula $2^{n/2}$ where n comprises the integers from minus eight to eight.

Hypothetical Pairwise Comparison Matrix (PCM) reflecting a given Priority Vector (PV)

$$\begin{bmatrix} w_1/w_1 & w_1/w_2 & w_1/w_3 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & w_2/w_3 & \cdots & w_2/w_n \\ w_3/w_1 & w_3/w_2 & w_3/w_3 & \cdots & w_3/w_n \\ \vdots & \vdots & \vdots & & \vdots \\ w_n/w_1 & w_n/w_2 & w_n/w_3 & \cdots & w_n/w_n \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} nw_1 \\ nw_2 \\ nw_3 \\ \vdots \\ nw_n \end{bmatrix}$$

SIMULATION CONCEPT

with respect to the GOAL:

$$
\begin{array}{c}
 \\ C1 \\ C2 \\ C3 \\ C4
\end{array}
\begin{array}{c}
\begin{array}{cccc} C1 & C2 & C3 & C4 \end{array} \\
\begin{bmatrix}
1 & 1.4 & 3.5 & 1.16667 \\
0.714286 & 1 & 2.5 & 0.833333 \\
0.285714 & 0.4 & 1 & 0.333333 \\
0.857143 & 1.2 & 3 & 1
\end{bmatrix}
\end{array}
\quad
\begin{array}{c}
H_G PV_C \\
\begin{bmatrix}
0.35 \\ 0.25 \\ 0.10 \\ 0.30
\end{bmatrix}
\end{array}
$$

SIMULATION CONCEPT

with respect to criteria C1–C2:

$$
\begin{array}{c} \\ A1 \\ A2 \\ A3 \\ A4 \end{array}
\begin{array}{c} \begin{array}{cccc} A1 & A2 & A3 & A4 \end{array} \\
\begin{bmatrix} 1 & 1.4 & 2.33333 & 1.4 \\ 0.714286 & 1 & 1.66667 & 1 \\ 0.428571 & 0.6 & 1 & 0.6 \\ 0.714286 & 1 & 1.66667 & 1 \end{bmatrix} \end{array}
\quad
\begin{array}{c} \mathrm{H}_{\mathrm{C2}}^{\mathrm{C1}}\mathrm{PV}_{\mathrm{A}} \\ \begin{bmatrix} 0.35 \\ 0.25 \\ 0.15 \\ 0.25 \end{bmatrix} \end{array}
$$

SIMULATION CONCEPT

with respect to criteria C3–C4:

$$
\begin{array}{c} \\ A1 \\ A2 \\ A3 \\ A4 \end{array}
\begin{array}{c} \begin{array}{cccc} A1 & A2 & A3 & A4 \end{array} \\ \begin{bmatrix} 1 & 0.666667 & 0.285714 & 0.25 \\ 1.5 & 1 & 0.428571 & 0.375 \\ 3.5 & 2.33333 & 1 & 0.875 \\ 4 & 2.66667 & 1.14286 & 1 \end{bmatrix} \end{array}
\qquad
\begin{array}{c} \mathrm{H}_{\mathrm{C4}}^{\mathrm{C3}}\mathrm{PV}_{\mathrm{A}} \\ \begin{bmatrix} 0.10 \\ 0.15 \\ 0.35 \\ 0.40 \end{bmatrix} \end{array}
$$

where $\mathrm{H_G PV_A}$, $\mathrm{H_{C2}^{C1} PV_A}$, $\mathrm{H_{C4}^{C3} PV_A}$ denote partial hypothetical PV in the model.

SIMULATION CONCEPT

After standard AHP synthesis, the hypothetical total PV (HTPV) is obtained HTPV=[0.25, 0.21, 0.23, 0.31]^T.

Next, we are going to perturb every PCM in the presented framework. The scenario assumes application of the rounding procedure and also takes into account the obligatory assumption in conventional AHP applications i.e. the PCM reciprocity condition. In such cases, only judgments from the upper triangle of a given PCM are taken into account
and those from the lower triangle are replaced by the inverses of the former. Then, on the bases of such PCMs and with application of the REV we compute their respective partial PVs (PPV_{REV}). Finally we compute the total calculated priority vector ($TCPV_{REV}$) for the exemplary model of the AHP.

Perturbed Pairwise Comparison Matrix (PPCM) and Priority Vector derived from PPCM

$$\begin{bmatrix} x_1/x_1 & x_1/x_2 & x_1/x_3 & \cdots & x_1/x_n \\ x_2/x_1 & x_2/x_2 & x_2/x_3 & \cdots & x_2/x_n \\ x_3/x_1 & x_3/x_2 & x_3/x_3 & \cdots & x_3/x_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_n/x_1 & x_n/x_2 & x_n/x_3 & \cdots & x_n/x_n \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = \lambda_{\max} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix}$$

SIMULATION CONCEPT

with respect to the GOAL:

$$\begin{array}{c} \\ C1 \\ C2 \\ C3 \\ C4 \end{array} \begin{array}{c} \begin{array}{cccc} C1 & C2 & C3 & C4 \end{array} \\ \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1/3 & 1/2 & 1 & 1/3 \\ 1 & 1 & 3 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} PPV_{REV}^{G} \\ \begin{bmatrix} 0.304999 \\ 0.276859 \\ 0.113143 \\ 0.304999 \end{bmatrix} \end{array}$$

SIMULATION CONCEPT

with respect to criteria C1–C2:

$$
\begin{array}{c} \\ A1 \\ A2 \\ A3 \\ A4 \end{array}
\begin{array}{c} \begin{array}{cccc} A1 & A2 & A3 & A4 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1/2 & 1/2 & 1 & 1/2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \end{array}
\qquad
\begin{array}{c} \mathrm{PPV}_{\mathrm{REV}}^{\mathrm{C1C2}} \\ \begin{bmatrix} 0.285714 \\ 0.285714 \\ 0.142857 \\ 0.285714 \end{bmatrix} \end{array}
$$

SIMULATION CONCEPT

with respect to criteria C3–C4:

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \\ A4 \end{array} \begin{array}{c} \begin{array}{cccc} A1 & A2 & A3 & A4 \end{array} \\ \begin{bmatrix} 1 & 1/2 & 1/4 & 1/4 \\ 2 & 1 & 1/2 & 1/3 \\ 4 & 2 & 1 & 1 \\ 4 & 3 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \mathrm{PPV}_{\mathrm{REV}}^{\mathrm{C3C4}} \\ \begin{bmatrix} 0.0887547 \\ 0.1611320 \\ 0.3550190 \\ 0.3950950 \end{bmatrix} \end{array}$$

SIMULATION CONCEPT

After standard AHP synthesis, the following result is obtained $TCPV_{REV}=[0.2034, 0.2336, 0.2316, 0.3315]^T$ which is different from $HTPV=[0.25, 0.21, 0.23, 0.31]^T$. Comparing HTPV with its estimate $TCPV_{REV}$ we can compute earlier mentioned performance measures i.e. MSRC, MPCC and MAAD which reflect estimation quality of the REV. For the presented example we have SRCC=0.2, PCC=0.8142, MAD=0.023325.

Noticeably, in this way we can compare estimation quality of any priority deriving method available for AHP.

SELECTED ESTIMATION TECHNIQUES FOR THE AHP

The Prioritization Procedure	Formula for the Prioritization Procedure
Simple Normalized Column Sum – SNCS –	$w_{i(SNCS)} = \frac{1}{n}\sum_{j=1}^{n}\left(a_{ij} \Big/ \sum_{k=1}^{n} a_{kj}\right)$
Logarithmic Least Squares Method – LLSM –	$w_{(LLSM)} = \min \sum_{i=1}^{n}\sum_{j=1}^{n} \ln^2\left(a_{ij}\frac{w_j}{w_i}\right)$
Right Principal Eigenvector – REV –	$w = \lim_{k \to \infty}\left(\frac{A^k \times e}{e^T \times A^k \times e}\right)$
Logarithmic Utility Approach – LUA –	$w_{(LUA)} = \min \sum_{i=1}^{n} \ln^2\left(\sum_{j=1}^{n}\frac{a_{ij}w_j}{nw_i}\right)$

SELECTED PROBABILITY DISTRIBUTIONS OF PERTURBATION FACTOR (e)

Additionally, in order to reflect somehow human judgment imperfections we can implement to the presented scenario random errors i.e.

$$x_{ij} = e_{ij} w_{ij}$$

Usually recommended for (e) are such types of probability distributions (PD) as: *gamma*, *log-normal*, *truncated normal*, or *uniform*. Apart from these most popular, one can find applications of the PD: *Couchy, Laplace*, and *triangle* or *beta.*

EXAMINATION RESULTS – (1)

Performance measures	Number of alternatives (n)				
	3	4	5	6	7
MPCC	1	0.999999	0.999997	0.999993	0.999991
MSRC	1	1	1	1	1
Performance measures	Number of alternatives (n)				
	8	9	10	11	12
MPCC	0.999989	0.999991	0.999987	0.999997	0.999988
MSRC	1	1	1	1	1

Simulation results of comparative studies concerning the LUA and the REV for 1000 randomly generated single RTPCMs

EXAMINATION RESULTS – (2)

In second examination we generate randomly one hundred ‘true’ PVs (uniform probability distribution) of the assigned size *n* x 1 what gives us one hundred PCMs created on the bases of each of them (perfectly consistent matrices *A(w)* of the size *n* x *n*). Then, we perturb one hundred times each PCM generated in this way following the earlier presented formula, where e_{ij} has uniform probability distribution within certain assigned intervals: [0.8, 1.2], [0.5, 1.5] and [0.2, 1.8]. Next, we calculate PVs: w_{LUA} and w_{REV} on the bases of these matrices with the application of the LUA and the REV, respectively.

EXAMINATION RESULTS – (2)

Then, we compare such obtained results with the values of original 'true' PVs. Thus, we analyze ten thousands cases for each given n and the size of the interval for e_{ij}.

In order to evaluate the performance of the LUA and the REV, we calculate earlier mentioned measures i.e. MSRC and MPCC between approximated and 'true' PVs. Additionally, we compute also other, known from literature performance measures, i.e. MAAD and average Root-Mean-Square-Deviation (aRMSD)

EXAMINATION RESULTS – (2)

Performance measures		MSRC	MPCC	aRMSD	MAAD
FRPCM	REV	0.97664	0.998604	0.00991015	0.00820928
	LUA	0.97664	0.998604	0.00991016	0.00820928

Performance evaluations of the LUA and the REV for n=4 and $e_{ij} \in [0.8, 1.2]$ for 10 000 cases

EXAMINATION RESULTS – (2)

Performance measures		MSRC	MPCC	aRMSD	MAAD
FRPCM	REV	0.89540	0.954185	0.0525000	0.0433404
	LUA	0.89550	0.954398	0.0524077	0.0432603

Performance evaluations of the LUA and the REV for n=4 and $e_{ij} \in [0.2, 1.8]$ for 10 000 cases

EXAMINATION RESULTS – (2)

Performance Measures		MSRC	MPCC	aRMSD	MAAD
FRPCM	REV	0.959014	0.989865	0.0134238	0.00982693
	LUA	0.959143	0.989878	0.0134200	0.00982263

Performance evaluations of the LUA and the REV for $n=8$ and $e_{ij} \in [0.5, 1.5]$ for 10 000 cases

EXAMINATION RESULTS – (2)

Performance Measures		MSRC	MPCC	aRMSD	MAAD
FRPCM	REV	0.990752	0.998930	0.00295574	0.00207079
	LUA	0.990753	0.998930	0.00295575	0.00207077

Performance evaluations of the LUA and the REV for $n=12$ and $e_{ij} \in [0.8, 1.2]$ for 10 000 cases

EXAMINATION RESULTS – (2)

Performance Measures		MSRC	MPCC	aRMSD	MAAD
FRPCM	REV	0.941472	0.962850	0.0174951	0.01202070
	LUA	0.942608	0.963834	0.0173122	0.01187650

Performance evaluations of the LUA and the REV for *n=12* and $e_{ij} \in [0.2, 1.8]$ for 10 000 cases

EXAMINATION RESULTS – (3)

Performance measures		MSRC	MPCC	MAAD
FRPCM	REV	0.823344	0.951072	0.0236221
	LUA	0.828300	0.953650	0.0231612

Performance evaluations of the LUA and the REV for 10 000 cases within uniformly drawn AHP framework:

$$n_c, n_a \in \{5, \ldots, 9\} \text{ and } e_{ij} \in [0.1, 1.9]$$

EXAMINATION RESULTS – (3)

Performance Measures		MSRC	MPCC	MAAD
FRPCM	REV	0.991800	0.99987	0.00135085
	LUA	0.991800	0.99987	0.00135085

Performance evaluations of the LUA and the REV for 10 000 cases within uniformly drawn AHP framework:

$$n_c, n_a \in \{5, \ldots, 9\} \text{ and } e_{ij} \in [0.9, 1.1]$$

EXAMINATION RESULTS – (3)

Performance measures		MSRC	MPCC	MAAD
FRPCM	REV	0,891378	0,980005	0,01485830
	LUA	0,893134	0,980570	0,01472490

Performance evaluations of the LUA and the REV for 10 000 cases within uniformly drawn AHP framework: n_c, $n_a \in \{4, \ldots, 12\}$

and $e_{ij} \in [0.5, 1.5]$ with gamma probability distribution

EXAMINATION RESULTS – (3)

Performance measures		MSRC	MPCC	MAAD
FRPCM	REV	0,917239	0,985688	0,01280100
	LUA	0,917174	0,985835	0,01275180

Performance evaluations of the LUA and the REV for 10 000 cases within uniformly drawn AHP framework: n_c, $n_a \in \{4, \ldots, 12\}$

and $e_{ij} \in [0.5, 1.5]$ with log-normal probability distribution

EXAMINATION RESULTS – (3)

Performance measures		MSRC	MPCC	MAAD
FRPCM	REV	0,977246	0,998918	0,00359482
	LUA	0,977246	0,998919	0,00359457

Performance evaluations of the LUA and the REV for 10 000 cases within uniformly drawn AHP framework: n_c, $n_a \in \{4, \ldots, 12\}$

and $e_{ij} \in [0.5, 1.5]$ with truncated-normal probability distribution

EXAMINATION RESULTS – (3)

Performance measures		MSRC	MPCC	MAAD
FRPCM	REV	0,955477	0,995495	0,00691381
	LUA	0,955481	0,995503	0,00691094

Performance evaluations of the LUA and the REV for 10 000 cases within uniformly drawn AHP framework: n_c, $n_a \in \{4,\ldots, 12\}$

and $e_{ij} \in [0.5, 1.5]$ with uniform probability distribution

CONCLUSIONS

$\Rightarrow$ both procedures taken into consideration in this research are very competitive;

$\Rightarrow$the REV and the LUA comparative evaluation leads to a conclusion that the REV, in some circumstances, may provide even worse than the LUA, estimation results of decision makers true priorities;

$\Rightarrow$ the phenomena observed in this research concerns the entire AHP model (not only single PCM).

THANK YOU!